ELEMENTS OF APPLIED MATHEMATICS

Problem set 1 - Uncoupled Linear Systems

Problem 1 Find the general solution and draw the phase portrait for the following linear systems:

(a) $\dot{x}_1 = x_1, \quad \dot{x}_2 = x_2,$

(b) $\dot{x}_1 = x_1, \quad \dot{x}_2 = 2x_2,$

(c) $\dot{x}_1 = x_1, \quad \dot{x}_2 = 3x_2,$

(d) $\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1$ (*Hint: note that $x_1^2 + x_2^2 =$ constant on the solution curves*),

(e) $\dot{x}_1 = -x_1 + x_2, \quad \dot{x}_2 = -x_2.$

Problem 2 Find the general solution and draw the phase portraits for the following three-dimensional linear systems:

(a) $\dot{x}_1 = x_1, \quad \dot{x}_2 = x_2, \quad \dot{x}_3 = x_3,$

(b) $\dot{x}_1 = -x_1, \quad \dot{x}_2 = -x_2, \quad \dot{x}_3 = x_3,$

(c) $\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = x_3.$

Hint: In (c), show that the solution curves lie on right circular cylinders perpendicular to the x_1, x_2 plane. Identify the stable and unstable subspaces in (a) and (b). The x_3-axis is the stable subspace in (c) and the x_1, x_2 plane is called the center subspace in (c).

Problem 3 Find the general solution of the linear system

$$\dot{x}_1 = x_1, \qquad \dot{x}_2 = ax_2$$

where a is a constant. Sketch the phase portraits for $a = -1$, $a = 0$ and $a = 1$ and notice that the qualitative structure of the phase portrait is the same for all $a < 0$ as well as for all $a > 0$, but that it changes at the parameter value $a = 0$ called *a bifurcation value*.

Problem 4 Find the general solution of the linear system $\dot{x} = Ax$ when A is the $n \times n$ diagonal matrix $A = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. What condition on the eigenvalues $\lambda_1, \ldots, \lambda_n$ will guarantee that $\lim_{t\to\infty} x(t) = 0$ for all solutions $x(t)$ of this system?

Problem 5 What is the relationship between the vector fields defined by

$$\dot{x} = Ax \qquad \text{and} \qquad \dot{x} = kAx$$

where k is a non-zero constant? (Describe this relationship both for k positive and k negative.)

Problem 6

(a) If $u(t)$ and $v(t)$ are solutions of the linear system $\dot{x} = Ax$, prove that for any constants a and b, $w(t) = au(t) + bv(t)$ is a solution.

(b) For $A = \mathrm{diag}(1, -2)$ find solutions $u(t)$ and $v(t)$ of $\dot{x} = Ax$ such that every solution is a linear combination of $u(t)$ and $v(t)$.