ELEMENTS OF APPLIED MATHEMATICS

Problem set 5 - Jordan form and stability theory, pt. 2

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1. Draw phase diagram for the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

2. Using Jordan canonical form:

   (a) Compute $\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}^{30}$.

   (b) Solve the equation $X^2 = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ in $M_2(\mathbb{C})$.

3. Prove that for any $A \in M_n(\mathbb{C})$ and $k \in \mathbb{Z}_+$ the equation $X^k = A$ has a solution.

   (Hint: find the Jordan canonical form of $J_m(\lambda)^k$.)

4. Solve the system

$$\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

   Find the stable, unstable and center subspaces for this system and sketch the phase portrait. For $x_0 \in E^c$, show that the sequence of points $x_n = e^{An}x_0 \in E^c$; similarly, for $x_0 \in E^s$ or E^u, show that $x_n \in E^s$ or E^u respectively.

5. Let A be an $n \times n$ nonsingular matrix and let $x(t)$ be the solution of the initial value problem $\dot{x} = Ax$ with $x(0) = x_0 \in \mathbb{R}^n$. Show that if $x_0 \in E^s \setminus \{0\}$ then $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} |x(t)| = \infty$.

6. Prove that for any power series f and any parameter $t \in \mathbb{C}$:

$$f(J_k(\lambda)t) = \begin{bmatrix} f(\lambda) & t \cdot f'(\lambda) & t^2 \cdot \frac{f''(\lambda)}{2} & \dots & t^{n-1} \cdot \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & t \cdot f'(\lambda) & \dots & t^{n-2} \cdot \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda) & t \cdot f'(\lambda) \\ 0 & 0 & 0 & 0 & f(\lambda) \end{bmatrix}.$$