Beauville – Complex algebraic surfaces. Solutions

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Part of solutions is "stolen" from Fumiaki Suzuki's "Solutions Of Exercises In Complex Algebraic Surfaces".

II Birational maps

II (1) Let P be a point of multiplicity m on C. Let $\varepsilon : \widetilde{S} \to S$ be the blow-up at P. Then $\widetilde{C} = \varepsilon^* C - mE$ and thus by genus formula:

$$\begin{aligned} p_{ar}(\widetilde{C}) &= 1 + \frac{1}{2}\widetilde{C}.(\widetilde{C} + K_{\widetilde{S}}) \\ &= 1 + \frac{1}{2}(\varepsilon^* C - mE).(\varepsilon^* C - mE + \varepsilon^* K_S + E) \\ &= 1 + \frac{1}{2}(C^2 + K_S.C - m^2 + m) \\ &= p_{ar}(C) - \frac{1}{2}m \cdot (m-1). \end{aligned}$$

Thus blowing up strictly decreases the arithmetic genus and after finitely many steps our curve will be smooth.

II (2) (a)

- ad. equality $m = \widehat{C}.E$:
  $\widehat{C} = \pi^* C - m$. Thus $m.\widehat{C} = E.\pi^* C - mE^2 = m$.
- ad. inequality $m_x(\widehat{C} \cap E) \geqslant m_x(\widehat{C})$:
  let f, g be the local equations of $\widehat{C}$ and E at x. Let $M = m_x(\widehat{C} \cap E)$; then $f \in \mathfrak{m}_x^M$. In particular, $(f, g) \subset \mathfrak{m}_x^M$ and:
  $$m_x(\widehat{C} \cap E) = \dim_k \mathcal{O}_{\widehat{S},x}/(f,g) \geqslant \dim_k \mathcal{O}_{\widehat{S},x}/\mathfrak{m}_x^M = M.$$

(b) Let r, s be the multiplicities of C and C' at p, respectively. Recall that if π is a blow-up at p, $\pi^* C = \widetilde{C} + rE$, $\pi^* C' = \widetilde{C}' + sE$. Thus:

$$\widetilde{C}.\widetilde{C}' = (\pi^* C - rE).(\pi^* C' - sE) = \pi^* C.\pi^* C' - r\pi^* C.E - sE.\pi^* C' + rsE.E = C.C' - 0 - 0 + rs.$$

Keep blowing up the surface at all intersection points of C and C' over p until C and C' do not meet transversally at all those points. Let $C_n := \widetilde{C}_{n-1}$, $C_n' := \widetilde{C}_{n-1}'$ be the images under those blow-ups. For $n \gg 0$, we will obtain that C_n and C_n' will meet transversally at all points above p and C_n, C_n' do not posses any multiple points above p. Thus $C_n.C_n' = \sum_x 1 = \sum_x m_x(C_n) \cdot m_x(C_n')$ (where the sum is take over all $x \in C_n \cap C_n'$ above p) and

$$m_p(C \cap C') = \sum_i r_i s_i + C_n.C_n' = \sum_{p \in C \cap C'} m_p(C) \cdot m_p(C')$$

(the last sum including infinitely near points).

(c) Recall that in 2.1 we showed that after a blow-up with center in a point of multiplicity m we obtain a curve $\widetilde{C}$ with arithemtic genus:

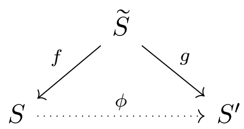
$$p_{ar}(\widetilde{C}) = p_{ar}(C) - \frac{1}{2}m \cdot (m-1).$$

Thus after finitely many blow-ups we arrive at normalization N, whose genus satisfies:

$$p_{ar}(N) = p_{ar}(C) - \sum_i \frac{1}{2}m_i \cdot (m_i - 1),$$

which ends the proof.

II (3) (a) By Corollary II.12 (elimination of indeterminacy + universality property of blow-up) we have a diagram:



where f, g are compositions of blow-ups. By blowing up $\widetilde{S}$ further at the non-smooth points of the strict transform of C, we can WLOG assume that $\widetilde{C}$, the strict transform of C on $\widetilde{S}$, is smooth. Then $g(\widetilde{C})$ is a point, and thus $\widetilde{C}^2 = -1$ and $\widetilde{C} \cong \mathbb{P}^1$. Thus C is birational to $\mathbb{P}^1$. Moreover, it is straightforward that:

$$\widetilde{C} = f^*C - \sum_i m_i E_i - \sum_{i=1}^{n} E_i'$$

where E_i are the exceptional divisors coming from blow-ups of singular points of C and E_i' are the exceptional divisors coming from blow-ups of smooth points of C (possibly including infinitely near points), i.e.

$n = \#\{$number of blow-ups in f, centered at smooth points of C (possibly including infinitely near points)$\}$.

Thus $-1 = \widetilde{C}^2 = C^2 - \sum_i m_i^2 - n$. Moreover, if C is smooth, $n > 0$ (since otherwise f would be an isomorphism and ϕ would be defined on whole C).

(b) Let $C^2 = \sum_i m_i^2 - 1 + n$ for $n \geqslant 0$ ($n > 0$ if C is smooth). Let $f : \widetilde{S} \to S$ be the blow-up of S at all singular points of C (including infinitely near points) and arbitrary n smooth points. Then $\widetilde{C}$ is smooth, $\widetilde{C} = f^*C - \sum_i m_i E_i - \sum_{i=1}^n E_i'$ and thus $\widetilde{C}^2 = C^2 - \sum_i m_i^2 - n = -1$ and thus by Castelnuovo criterion there exists a morphism $g : \widetilde{S} \to S'$ such that $g(\widetilde{C})$ is a point. Thus it suffices to take $\phi = g^{-1} \circ f$.

III Ruled surfaces

III (1) Recall that $F^2 = 0$ and $\widetilde{F} = \pi^*F - E$ (where $\pi : \widetilde{S} \to S$ is a blow-up of S on an arbitrary point of F). Thus $\widetilde{F}^2 = F^2 + E^2 = F^2 - 1 = -1$ and we can contract $\widetilde{F}$ by the Castelnuovo criterion: $\widetilde{S} \to S'$.

III (2)

Errata: a point of $s \in \mathbb{P}(E)$ over $x \in C$ corresponds to a morphism:

$$E^\vee \to i_{x,*}\mathbb{C} \to 0.$$

E' should be defined by the exact sequence:

$$0 \to (E')^\vee \to E^\vee \to i_{x,*}\mathbb{C} \to 0.$$

Recall: here we define $\mathbb{P}(E) := \operatorname{Spec}\operatorname{Sym} E^\vee$, thus the points $s \in \mathbb{P}(E)$ over $x \in C$ correspond to:

- elements of $\mathbb{P}(E_x \otimes \kappa(x))$,
- lines in the $\mathbb{C}$-vector space $E_x \otimes \kappa(x)$,
- morpshisms $E^\vee \to i_{x,*}\mathbb{C} \to 0$.

Note moreover that any morphism of vector bundles $f : E \to E'$ induces a rational map $\mathbb{P}(f) : \mathbb{P}(E) \dashrightarrow \mathbb{P}(E')$ – it is well-defined out of the set:

$$\{(x \in C, \xi \in \mathbb{P}(E_x \otimes \kappa(x))) : f(\xi) = 0 \text{ in } E_x' \otimes \kappa(x)\}$$

We start by proving that E' is a rank 2 vector bundle. Note that $(E')^\vee$ is locally free as a subsheaf of a locally free sheaf. Moreover $(E')^\vee \cong E^\vee$ out of x and $0 \to (E')^\vee_x \to E^\vee_x \to \mathbb{C} \to 0$. If $(E')^\vee_x$ was a free $\mathcal{O}_x$-module of rank $\leqslant 1$, then the quotient would contain a copy of $\mathcal{O}_x$ – contradiction. Hence $(E')^\vee_x$ is of rank 2 and E is a vector bundle of rank 2.

Let $h : E \to E'$ be the dual of the inclusion $(E')^\vee \to E^\vee$. We'll show that:

(A) $\mathbb{P}(h)$ is an isomorphism out of $F := p^{-1}(x)$,

Pf: By definition of E', $E|_U \cong E'|_U$, where $U := \mathbb{P}(E)\backslash F$. Therefore $\mathbb{P}(h)$ is an isomorphism out of F.

(B) $\mathbb{P}(h) : \mathbb{P}(E) \to \mathbb{P}(E')$ is defined out of s and contracts F to a point,

Pf: we only need to check it over x. Recall that $\mathbb{P}(h)$ is defined as

$$\mathbb{P}(E_x \otimes \kappa(x)) \ni \xi \mapsto [h(\xi)] \in \mathbb{P}(E'_x \otimes \kappa(x)),$$

i.e. it is well defined on a line $\xi \in E_x \otimes \kappa(x)$, unless $\xi \subset \ker(h_x \otimes \kappa(x))$. Recall that we have the exact sequence:

$$(E')^\vee_x \otimes \kappa(x) \xrightarrow{h^\vee} E^\vee_x \otimes \kappa(x) \to \mathbb{C} \to 0$$

or equivalently,

$$0 \to \xi_s \to E_x \otimes \kappa(x) \to E'_x \otimes \kappa(x)$$

where ξ_s is the line corresponding to s. Note that thus the dimension of the image of

$$I := im\left(E_x \otimes \kappa(x) \to E'_x \otimes \kappa(x)\right)$$

is one. Under h, every line in $E_x \otimes \kappa(x)$ goes either to 0 (if this line is ξ_s, i.e. if it is a point corresponding to s) or to I (if this line is not ξ_s). Thus $\mathbb{P}(h)$ is not defined at s and the image of $F\backslash\{s\}$ under h is the point $s' \in S'$ corresponding to the line:

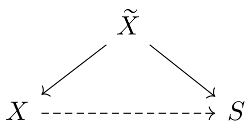
$$0 \to I \to E'_x \otimes \kappa(x).$$

(C) $\mathbb{P}(E')$ contains an "additional" rational line $p'^{-1}(x)$.

Pf: this is straightforward.

The properties (A), (B), (C) show that $\mathbb{P}(E') = S$.

III (3) By Corollary II.12, we can present the map $\phi : X \dashrightarrow S$ as:

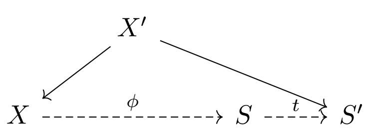


where $\widetilde{X} \to X$, $X \to S$ are compositions of isomorphisms and blow-ups and $\widetilde{X} \to X$ is composed of $n = n(\phi)$ blow-ups. Let $\varepsilon : \widetilde{X} \to X'$ be the last blow-up with the center P and exceptional divisor $E \subset \widetilde{X}$. Note that:

- the image of E in S is not a point – otherwise, the map $X' \dashrightarrow X \to S$ and its inverse would have a single indeterminancy point, which would contradict Lemma II.10,
- the image of E in S (denoted also E) is a fiber of $S \to C$. Indeed, it is a rational curve and the only rational curves S contains, are the fibers (here we use the assumption $C \neq \mathbb{P}^1$).
- $\widetilde{X} \to S$ must contain at least one blow-up on a point of (strict transform of) E_S. Indeed, otherwise E would have the same intersection number on $\widetilde{X}$ (which is -1, since it is an exceptional divisor of a blow-up) and on S (which is zero for any fiber).

  Say that this blow-up was with center on $s \in E$ and that the exceptional divisor (or rather its strict transform with respect to the next blow-ups) is $F \subset \widetilde{X}$.

Let $\widetilde{S}$ be the blow-up of S at s and let $t : S \dashrightarrow S'$ be the elementary transform of S at s. We want to show that we have the following diagram:



This will end the proof, since then

$$n(\phi \circ t) = \#\ (\text{number of blow-ups in } X' \to X) = n - 1.$$

Note firstly, that since $\widetilde{X} \to S$ contracts F, it must factor as $\widetilde{X} \to \widetilde{S} \to S$ (Proposition II.8). Now consider the birational map $\psi : X' \dashrightarrow \widetilde{X} \to \widetilde{S} \to S'$. Note that ψ is undefined at at most one point – P, the image of E. But there doesn't exist a curve $C \subset S'$ such that $\psi^{-1}(C) = P$ (otherwise, the strict transform of this curve on $\widetilde{X}$ would be E, but E is contracted on S'). Thus, by Lemma II.10, the map ψ is defined at P and we obtain a morphism $X' \to S'$. This ends the proof.

III (4) Note that $PGL(2, K) = Aut(\mathbb{P}(E_\xi))$, where ξ is the generic point of C. Thus, any $\varphi \in PGL(2, K)$ corresponds to $\varphi : \mathbb{P}(E_\xi) \to \mathbb{P}(E_\xi)$ and this may be extended to $\varphi : p^{-1}(U) \to p^{-1}(V)$ for some open sets $U, V \subset C$. Thus we obtain a map:

$$PGL(2, K) \to \mathrm{Aut}_b(S),$$

which is clearly injective. Choose a section $s : C \to S$. Then we have a map:

$$\mathrm{Aut}_b(S) \to \mathrm{Aut}(C), \qquad \Psi \mapsto p \circ \Psi \circ s$$

(note that $p \circ \Psi \circ s$ is a birational map $C \dashrightarrow C$ which easily implies – since C is smooth and projective – that it extends to an automorphism $C \to C$). We want to show that $\phi := p \circ \Psi \circ s$ satisfies $p \circ \Psi = \phi \circ p$ (cf. Remark III.16). This follows by $C \not\cong \mathbb{P}^1$. Indeed, note that for any fiber F, $\Psi(F)$ is also a fiber, since it is isomorphic to $\mathbb{P}^1$, and S doesn't contain rational curves other than fibers (here we use $C \not\cong \mathbb{P}^1$). Thus, since x and $s \circ p(x)$ lie in the same fiber, $\Psi(x)$ and $\Psi \circ s \circ p(x)$ lie also in the same fiber, i.e. $p \circ \Psi(x) = p \circ \Psi \circ s \circ p(x)$, i.e. $p \circ \Psi = \phi \circ p$.

Moreover, $\Psi \in \mathrm{Aut}_b(C)$ maps to $id \in \mathrm{Aut}(C)$ iff $p \circ \Psi = id$. But then, after replacing C by an open subset U, we obtain the commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^1 \times U & \xrightarrow{\Psi} & \mathbb{P}^1 \times U \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

i.e. $\Psi \in PGl(2, \mathcal{O}(U))$, i.e. Ψ comes from $PGl(2, K)$.

Fix $\phi \in \mathrm{Aut}(C)$. Suppose that $V \subset C$ is an open set and $U = \varphi^{-1}(V)$ and that U, V are small enough so that $p^{-1}(U) \cong \mathbb{P}^1 \times U$, $p^{-1}(V) \cong \mathbb{P}^1 \times V$. Let

$$\Psi := (id, \phi) : p^{-1}(U) \cong \mathbb{P}^1 \times U \to \mathbb{P}^1 \times V \cong p^{-1}(V)$$

– then $\Psi \in \mathrm{Aut}_b(S)$ and Ψ maps to ϕ. This proves the surjectivity and easily shows the splitting.

III (5)

Recall that a point $s \in S = \mathbb{P}(E)$ lying over $t \in C$ corresponds to a surjective morphism:

$$\varphi : E^\vee \to i_{t,*}\mathbb{C} \to 0.$$

By ex. III (2) we want to compute $E' = (\ker \varphi)^\vee$.

Suppose WLOG that s lies over $t = [0 : 1] \in \mathbb{P}^1$. Note that any surjective morphism $\varphi : E^\vee = \mathcal{O} \oplus \mathcal{O}(-n) \to i_{t,*}\mathbb{C}$ is of the form $a\pi_1 + b\pi_2$, where $(a, b) \in \mathbb{C}^2 \backslash \{0\}$ and

$$\pi_1 : \mathcal{O} \to i_{t,*}\mathcal{O}_t \to i_{t,*}(\mathcal{O}_t/\mathfrak{p}_t) \cong i_{t,*}\mathbb{C}$$

$$\pi_2 : \mathcal{O}(-n) \to i_{t,*}\mathcal{O}(-n)_t \to i_{t,*}(\mathcal{O}(-n)_t/\mathfrak{p}_t\mathcal{O}(-n)) \cong i_{t,*}\mathbb{C}.$$

Let φ correspond to (a, b). Note that for any quasicoherent sheaf $\mathcal{F} \to \widetilde{M}$ on $\mathrm{Proj}\, S$, the morphisms

$$\mathcal{F} \to i_{x,*}\mathcal{F}_x, \qquad \mathcal{F} \to i_{x,*}(\mathcal{F}_x/\mathfrak{p}_x\mathcal{F}_x)$$

correspond to homomorphisms

$$M \to M_{\mathfrak{p}}, \qquad M \to M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \cong Frac(M/\mathfrak{p})$$

of graded S-modules (where x corresponds to an ideal $\mathfrak{p}$ and $M_{\mathfrak{p}}$ denotes the **homogeneous localisation**, and $Frac$ – the homogeneous fraction field).

In our case, $S = \mathbb{C}[x,y]$ and π_1, π_2 come from homomorphisms

$$S \to S/\mathfrak{p} \qquad \text{and} \qquad S(-n) \to S(-n)/\mathfrak{p}$$

which we will also denote by π_1, π_2 (where $\mathfrak{p} = (y)$). Note that we can identify $S(-n)$ with $y^n S$ or $x^n S$. Moreover:

$$S(-n)_\mathfrak{p}/\mathfrak{p}S(-n)_\mathfrak{p} = x^n k[x,y]_\mathfrak{p}/yx^n k[x,y]_\mathfrak{p} \cong x^n k(x) \cong k(x) \cong k[x,y]_\mathfrak{p}/yk[x,y]_\mathfrak{p} \cong S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}.$$

Thus φ is given by:

$$\varphi(A(x,y), x^n B(x,y)) = A(x,0) + x^n B(x,0) = aA + bx^n B \pmod{y} \in k(x).$$

and $\ker\varphi = \widetilde{K}$, where:

$$K = \{(A, x^n B) \in S \oplus x^n S : aA + bx^n B \equiv 0 \pmod{y}.\}$$

We consider the following three cases:

1^o $a \neq 0$, $b = 0$.

In this case clearly $K \cong yS \oplus x^n S \cong S(1) \oplus S(-n)$, i.e. $\widetilde{K} = \mathcal{O}(1) \oplus \mathcal{O}(-n)$, i. e. $E' = \mathcal{O}(-1) \oplus \mathcal{O}(n)$, i.e.

$$S' = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(n)) \cong \mathbb{P}((\mathcal{O}(-1) \oplus \mathcal{O}(n)) \otimes \mathcal{O}(1)) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n+1)) = \mathbb{F}_{n+1}.$$

2^o $b \neq 0$.

In this case we have an isomorphism:

$$\begin{array}{rcl} yS \oplus x^n S & \cong & K \\ (yP, x^n Q) & \mapsto & \left(\frac{1}{a} yP - \frac{b}{a} x^n Q, x^n Q\right) \end{array}$$

i.e. $E' \cong \mathcal{O}(1) \oplus \mathcal{O}(n)$, i.e. $S' = \mathbb{P}(E') \cong \mathbb{P}(E' \otimes \mathcal{O}(-1)) = \mathbb{F}_{n-1}$.

Finally, we see that we have two cases:

- if s lies in the image of the section $\mathbb{P}^1 \to \mathbb{F}_n$ coming from the surjection $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathcal{O}$ (this section is denoted B in chapter IV), then $S' \cong \mathbb{F}_{n+1}$,
- if $s \notin B$, then $S' \cong \mathbb{F}_{n-1}$.

III (8)

(I guess that we want to classify ruled surfaces over C **up to C-homeomorphism**)

Lemma Let M be any compact oriented manifold of dimension 2. Then:

(a) We have an isomorphism of groups:

$$\deg : \text{ complex line bundles on } M \to \mathbb{Z}$$

that coincides with the degree function for smooth projective algebraic curves over $\mathbb{C}$

(b) We have an isomorphism of groups:

$$\deg \oplus \dim : \text{ complex vector bundles on } M \to \mathbb{Z} \oplus \mathbb{Z}.$$

Thus the ring of vector bundles on M is isomorphic to $\mathbb{Z}[x]/(x^2)$.

Pf:

(a)

By the above lemma, any vector bundle over C is isomorphic (as a complex vector bundle) to $\mathcal{O} \oplus \mathcal{O}(n)$ for $n = \deg E$ or equivalently to $\mathcal{O}(m) \oplus \mathcal{O}(m+\varepsilon)$ for $n = 2m + \varepsilon$, $\varepsilon \in \{0,1\}$. We have: $\mathbb{P}(\mathcal{O}(m) \oplus \mathcal{O}(m+\varepsilon))$ is C-isomorphic to $\mathbb{P}((\mathcal{O}(m) \oplus \mathcal{O}(m+\varepsilon)) \otimes \mathcal{O}(-m)) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\varepsilon))$. It suffices to show now that $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\varepsilon))$ and $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\varepsilon))$ are not C-homeomorphic. ?????

III (10) (solution stolen from Suzuki's solutions)

We'll start by showing that S must contain uncountably many lines.

Lemma Let $k = \overline{k}$ be an uncountable field, and let X be a k-variety. Let $(Z_n)_n$ be a countable family of proper closed subschemes of X. Then $\bigcup_i Z_i(k) \neq X(k)$.
Proof – I method: (MO 73743) by shrinking X, we can assume that it is affine. By Noether normalization lemma, there exists a finite surjective morphism $p : X \to \mathbb{A}^m_k$. Let $Y_i := p(Z_i)$. Then $\mathbb{A}^m_k(k) = \bigcup_i Y_i(k)$. It suffices to show that this impossible by induction on m. For $m = 1$ this is straightforward. Note that $\mathbb{A}^m_k(k)$ has uncountably many hyperplanes. Take a hyperplane H such that $\forall_i\, H \neq Y_i$. Then $\forall_i\, H \not\subset Y_i$ and thus $H = \bigcup_i (Y_i \cap H)(k)$ is a union of proper closed subvarieties. This is impossible by induction hypothesis.
Proof – II method: (only for $k = \mathbb{C}$) the proof follows by using Baire categories theorem, since a complete metric space (we can e.g. embedd X in $\mathbb{P}^n$ to get a metrics) cannot be a countable union of nowhere dense sets.

We'll consider two cases:

- **Case I:** $q(S) \geqslant 1$

  Let $A := \mathrm{Alb}(S)$, $j : S \to A$ and note that $\dim A = q(S) \geqslant 1$. Note that $\mathrm{Alb}(\mathbb{P}^1) = pt$, so all the rational lines on S are contracted to points. Thus, if $\dim j(S) = 2$, then j would contract infinitely many curves to points. But this would contradict the following Lemma.

  Lemma Let $f : S \to S'$ be a surjective morphism of surfaces. Then f contract only finitely many lines.
  Proof: (cf. MSE, 3413803) By generic freeness, there is a closed subset Z such that for $s \in S' \backslash Z$, $\dim f^{-1}(s) = \dim S - \dim S' = 0$. Note that $f^{-1}(Z)$ is a closed set of dimension $\leqslant 1$ and all of the contracted curves are contained in it. But $f^{-1}(Z)$ has finitely many irreducible components! This ends the proof.

  Thus $\dim j(S) = 1$. By generic smoothness (???) at least one of the fibers of j must be isomorphic to $\mathbb{P}^1$ (???) and the proof follows by Noether–Enriques Theorem in this case.

- **Case II:** $q(S) = 0$.

  Let H be a very ample divisor. Consider for every $n \in \mathbb{N}$ the set $A_n := \{C - \text{rational curve} : C.H = n\}$. Then, by Pigeonhole Principle, there exists $n \in \mathbb{N}$ such that A_n contains infinitely many curves. By [Hartshorne, AG, ex. ???] the set A_n modulo numerical equivalence is finite and thus there exist $C_1, C_2 \in A_n$, $C_1 \cong C_2$. Thus, $C_1^2 = C_1.C_2 \geqslant 0$ (intersection product of two irreducible curves is the number of their intersection points, counted with multiplicities). But then we conclude that S is rational just as in the proof of Castelnuovo Theorem.

IV Rational surfaces

IV (1)
- **Step I:** $P = |h|$ is n-dimensional, i.e. $h^0(\mathcal{O}(h)) = n + 1$.

  Pf: note that $\mathcal{O}(h) \cong \mathcal{O}_{\mathbb{F}_n}(1)$ and thus $p_*\mathcal{O}(h) = \mathcal{E} = \mathcal{O} \oplus \mathcal{O}(n)$ and:

$$h^0(\mathcal{O}(h)) = h^0(\mathbb{P}^1, p_*\mathcal{O}(h)) = h^0(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(n)) = n + 1.$$

- **Step II:** $|h|$ is very ample on $U := \mathbb{F}_n \backslash B$

  Step II A: $h^1(h - f) = 0$

  Pf: note that $(h - f).f = 1$ and thus by [Hartshorne, AG, Lemma V.2.4] $H^1(\mathcal{O}(h - f)) \cong H^1(\mathbb{P}^1, p_*(\mathcal{O}(h - f)))$. But by projection formula:

$$p_*(\mathcal{O}(h - f)) = (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(n - 1)$$

  (f may defined as $p^*(\text{any point})$). Thus $h^1(h - f) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(n-1)) = 0$. Cf. also Hartshorne, pf. Theorem V.2.17., Case IV.

  Step II B: separating points $P \neq Q$, $P, Q \notin B$.

  - if P, Q are not on one fiber, we can take $b + nf \in |h|$ for fiber f containing P, but not Q. Then $P \in b + nf$, $Q \notin b + nf$.

- suppose that P, Q are in one fiber f. Note that $h.f = 1$ and thus the linear system $|h|$ restricted to $f \cong \mathbb{P}^1$ is very ample and we can find a divisor separating P and Q. But the restriction of $|h|$ to f, i.e. the map:

$$H^0(\mathcal{O}(h)) \to H^0(\mathcal{O}(h) \otimes \mathcal{O}_f)$$

is surjective. Indeed, the cokernel is $H^1(\mathcal{O}(h-f))$ (this follows from the exact sequence $0 \to \mathcal{O}(h-f) \to \mathcal{O}(h) \to \mathcal{O}(h) \otimes \mathcal{O}_f \to 0$) and this is zero by Step II A.

- **Step II C:** separating a points $P \notin B$ and a tangent vector $v \in T_P\mathbb{F}_n$.
  - let f be the fiber of P. If $v \notin T_P f$, then $b + nf \in |h|$, $P \in b + nf$, $v \notin T_P(b + nf)$.
  - If $v \in T_P f$, we can repeat the reasoning from Step II B.
- **Step III:** $|h|$ is base point free,

  Pf: since $|h|$ is very ample outside of B, we only have to check that for every $x \in B$ there exists $D \in |h|$ such that $x \notin D$. But we can take $D := h$, since $h.b = 0$.
- **Step IV:** the image of B via f is a single point p.

  Pf: it suffices to show that the image of $|h - b| \to |h|$, $D \mapsto D + b$ is of codimension 1 in $|h|$. Indeed, then for any $x \in B$, the hypersurface

$$f(x) := D \in |h| \text{ that contain } x \ \in |h|^\vee$$

  must be $|h - b|$. We have: $|h - b| = |nf|$ and we are left with computing $h^0(nf)$. Using:
  - the exact sequence $0 \to \mathcal{O}((m-1)f) \to \mathcal{O}(mf) \to \mathcal{O}_f \otimes \mathcal{O}(m) \to 0$ for $m \geqslant 0$,
  - $H^1(\mathcal{O}_{\mathbb{P}^1}(t)) = 0$ for $t < 0$,
  - $H^1(\mathcal{O}_{\mathbb{F}_n}) = q(\mathbb{F}_n) = 0$,

  one can show that $h^1(mf) = 0$ for every $m \geqslant 0$ and that $h^0(mf) = m$. Thus $h^0(nf) = n$ and $|h - b|$ is indeed of codimension 1 in $|h|$.
- **Step V:** $|h|$ cut to the section h is the linear system $\mathcal{O}_{\mathbb{P}^1}(n)$ on $\mathbb{P}^1$. Thus the image of h by f is the line $\mathbb{P}^1$ embedded via Veronese embedding.

  Pf: indeed, the degree of the divisor h cut to the section h is $h.h = n$.
- **Step VI:** $|h|$ cut to any fiber is the linear system $\mathcal{O}_{\mathbb{P}^1}(n)$ on $\mathbb{P}^1$. Thus the image of any fiber is a line through p.

  Pf: indeed, the degree of the divisor h cut to f is $h.f = 1$.
- **Summary**: f is well defined, an embedding out of B, contracts B to one point, $f(h)$ is $\mathbb{P}^1$ embedded via Veronese embedding and the image of any fiber is a line through p. Therefore $f(\mathbb{F}_n)$ must be a cone over $f(h)$.

IV (3) Choose any $n-1$ distinct points on S and let H be the hypersurface containing them. Then by Bezout theorem $H \cap S \leqslant \deg S \cdot \deg H = (n-2)$ or $S \subset H$. We clearly see that only the second possibility can hold.

V Castelnuovo's Theorem

V (1) Note that for any n, $-nK$ is ample and thus $H^0(nK) = 0$ for every $n \geqslant 0$ (trivial case of Kodaira vanishing). Thus $P_n = 0$ for every n and S is rational by Castelnuovo theorem. Let S_{min} be the minimal model of S. Then $S_{min} = \mathbb{P}^2$ or $S_{min} = \mathbb{F}_n$ for $n \neq 1$. Note that $g : S \to S_{min}$ is composition of r blow ups for some r, with exceptional divisors $E_1, \ldots, E_r$. Then $K_S = g^* K_{S_{min}} + \sum_i E_i$. Suppose to the contrary that $S_{min} = \mathbb{F}_n$ for $n \geqslant 2$. Then $K_{S_{min}} = -2h + (n-2)f$. Consider $\widehat{B}$, the strict transform of $B \sim h - nf$. Note that $\widehat{B} \sim g^*h - g^*nf - \sum_{i \in I} E_i$ (we sum over the exceptional divisors of blow-ups with center in B). On the other hand, since $-K_S$ is ample, by Nakai-Moscheizon criterion, $-K_S.\widehat{B} > 0$, i.e.:

$$-K_S.\widehat{B} = (2g^*h - (n-2)g^*f - \sum_i E_i).(g^*h - g^*nf - \sum_{i \in I} E_i) = 2n - 2n - (n-2) - \#I \leqslant 2 - n$$

which is non-positive.

Thus $S_{min} = \mathbb{P}^1 \times \mathbb{P}^1$ or $S_{min} = \mathbb{P}^2$.

Suppose that $S_{min} = \mathbb{P}^2$, i.e. S is $\mathbb{P}^2$ with r points blown. Then $K_S = \pi^* K_{S_{min}} + \sum_{i=1}^r E_i = -3L + \sum_{i=1}^r E_i$. Thus:

$$0 < (-K_S).E_i = -E_i^2 - \sum_{j \neq i} E_i.E_j = 1 - - \sum_{j \neq i} E_i.E_j,$$

which implies that $E_i.E_j = 0$ (i.e. the r points are not infinitely near points, but points of $\mathbb{P}^2$). Moreover:

$$0 < K_S^2 = 9 - r$$

which implies that $r \leqslant 8$. Suppose that the r points do not lie in general position, i.e. either $t \geqslant 3$ (e.g. $P_1, \ldots P_t$) of them lie on a common line M or $t \geqslant 6$ of them (e.g. $P_1, \ldots, P_t$) on a common cubic C. Then:

$$(-K_S).\widetilde{M} = (-K_S).(\pi^* M - \sum_{i=1}^t E_i) = (3L - \sum_{i=1}^r E_i).(\pi^* M - \sum_{i=1}^t E_i) = 3L.M - t = 3 - t \leqslant 0$$

or

$$(-K_S).\widetilde{C} = (-K_S).(\pi^* C - \sum_{i=1}^t E_i) = (3L - \sum_{i=1}^r E_i).(\pi^* M - \sum_{i=1}^t E_i) = 3L.C - t = 6 - t \leqslant 0$$

– contradiction. Thus in this case S is isomorphic to $\mathbb{P}^2$ with $r \leqslant 8$ points in general position blown.

Suppose now that $S_{min} = \mathbb{P}^1 \times \mathbb{P}^1$.

????

V (3)

(This solution is stolen from Suzuki)

Step I: the group $\{\varphi \in \operatorname{Aut} \mathbb{P}^n : \varphi(S) = S\}$ is finite.

Proof:

Lemma: Suppose that an algebraic group G acts on a variety S. Then the function:

$$s \mapsto \dim Gs$$

is lower-semicontinuous. In particular, for s in a dense open subset $\dim Gs = \max\{\dim Gx : x \in S\}$.
Pf: consider the diagram:

$$\begin{array}{ccc} (G \times_S S) \times_{S \times S} S & \xrightarrow{p} & S \\ \downarrow & & \downarrow \Delta \\ G \times S & \xrightarrow{q} & S \times S \end{array}$$

where $q : G \times S \to S \times S$, $q(g, s) = (gs, s)$. Then one easily checks that $\operatorname{Stab}(s) \cong p^{-1}(S)$ and thus:

$$\dim Gs = \dim G - \dim \operatorname{Stab}(s) = \dim G - \dim p^{-1}(S).$$

It suffices to note that the dimension of the fiber is upper-semicontinuous.

Let G be the identity component of the algebraic group:

$$\{\varphi \in \operatorname{Aut} \mathbb{P}^n : \varphi(S) = S\}$$

Suppose to the contrary that $\dim G > 0$. Note that the orbits of action of G on S are intersections of linear subspaces of $\mathbb{P}^n$ with S. Moreover, they are connected, since G is. Let $m = \max\{\dim Gx : x \in S\}$ and consider the following possibilities:

1^o $m = 0$. In this case, Gs is a connected set of dimension 0, hence $Gs = \{s\}$. ????

2^o $m = 1$. Then S is covered by rational curves (since G acts linearly on S????) and thus by exercise III.10, S is ruled, contradiction.

3^o $m = 2$. Then Gs is a dense open subset of S and (since the preimage of G under $\mathrm{Gl}(n,\mathbb{C}) \to \mathrm{PGl}(n,\mathbb{C})$ is a linear group and any linear group over a perfect infinite field is unirational, cf. Springer, 13.3.10. Corollary) S is unirational. Thus, by Corollary V.5, S is rational, contradiction.

Step II: $\mathrm{Aut}\, S$ is as claimed.

Proof: Recall that $\mathrm{Aut}(S)$ (for S – projective) is a projective variety and thus $\mathrm{Aut}^0(S)$ is an abelian variety. Let H be a very ample divisor associated to the embedding $S \subset \mathbb{P}^n$. Consider the morphism

$$\Phi : \mathrm{Aut}^0(S) \to \mathrm{Pic}(S), \qquad \varphi \mapsto \varphi^* H - H.$$

Note that since $\mathrm{Aut}^0(S)$ is connected, its image must lie in the connected component of $\mathrm{Pic}(S)$, i.e. $\mathrm{Pic}^0(S) \cong \mathrm{Alb}(S)$, which is of dimension q. Now, $\mathrm{Aut}^0(S) \to \mathrm{Pic}^0(S)$ is a morphism between abelian varieties which maps identity to identity, and thus it is an homomorphism. Thus $\Phi(\mathrm{Aut}^0(S))$ is an abelian variety of dimension $\leqslant q$. Moreover the kernel of Φ consists of those φ, which commute with $\phi_{|H|}$. We can extend each such φ to an automorphism of $\mathbb{P}^n = |H|^\vee$, since φ induces an isomorpshism of $|H|$. Thus:

$$\ker \Phi = \{\varphi \in \mathrm{Aut}\, \mathbb{P}^n : \varphi(S) = S\}$$

is finite by Step I and the map $\mathrm{Aut}^0(S) \to \mathrm{im}\Phi$ is an isogeny of abelian varieties, i.e. $\mathrm{Aut}^0(S)$ is an abelian variety of dimension $\leqslant q$. This ends the proof.

V (4) Note that we can identify $H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ with $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(n))$. Note that any element (φ, c) of

$$T := \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(n)) \rtimes \mathbb{C}^*$$

corresponds to an automorphism $\Gamma_{(\varphi,c)}$ of the bundle $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$:

$$(x, y) \mapsto (cx, y + \varphi(x))$$

that fixes $0 \oplus \mathcal{O}(n) \subset \mathcal{E}$. In this way we obtain a morphism

$$\mathbb{F}_n = \mathbb{P}(\mathcal{E}) \xrightarrow{\Gamma^*_{(\varphi,c)}} \mathbb{P}(\mathcal{E}) = \mathbb{F}_n,$$

which easily provides us a homomorphism $T \to \mathrm{Aut}\, \mathbb{F}_n$.

Let $\Gamma \in \mathrm{Aut}\, \mathbb{F}_n$ be now arbitrary. Note that $\Gamma(b) = b$ (since b is the unique curve on $\mathbb{F}_n$ with negative self-intersection) and thus we obtain a morphism $\mathrm{Aut}\, \mathbb{F}_n \to \mathrm{Aut}\, b = \mathrm{Aut}\, \mathbb{P}^1 = \mathrm{PGl}(2, \mathbb{C})$.

Note that the map $\mathrm{Aut}\, \mathbb{F}_n \to \mathrm{Aut}\, b$ is onto, since it has a natural section – $\varphi \in \mathrm{Aut}\, b = \mathrm{Aut}\, \mathbb{P}^1$ maps to $\mathbb{P}(\mathcal{E}) \xrightarrow{\varphi^*} \mathbb{P}(\varphi^* \mathcal{E}) \cong \mathbb{P}(\mathcal{E})$ (???).

????

V (5)

Erratum: I don't think the hint with $D \mapsto D + (\delta.D)\delta$ is useful, since this is an involution.

(Second part of the solution is based on R. Friedman, Algebraic Surfaces and Holomorphic Vector Bundles, Ch. 5, Prop. 22)

If S contains infinitely many lines then it is rational:

Let $f : S \to S_{min}$ be the morphism to the minimal model and suppose that it is a composite of n blow ups with exceptional divisors $E_1, \ldots, E_n$. Note that f contracts finitely many curves $C_1, \ldots, C_m$. Let C be an exceptional curve different from the E_i's and C_i's. Then $f(C)$ is a rational curve with $f(C)^2 \geqslant 0$ (since each blow-up decreases the self-intersection and $f(C)^2 \neq -1$ – S_{min} doesn't contain exceptional curves). Consider the morphism $\alpha : S_{min} \to \mathrm{Alb}(S_{min})$. Note that it contracts infinitely many curves (all the $f(C)$'s) and thus (cf. exercise III (10), Lemma), $\dim \alpha(S_{min}) < 2$.

We conclude that S_{min} is ruled or rational as in the end of the proof of Theorem V.19:

We have $f(C)^2 \geqslant 0$, since $C^2 = -1$ and each blow-up increases this value. Note that each blow-up descreases $K_S.C = -1$ and that each blow-up decreases this value, thus $K_{S_{min}}.f(C) \leqslant -2$. Let F be the fiber of α containing $f(C)$. Then Lemma III.19 shows that $F = r \cdot f(C)$. Then $F^2 = 0$ and thus $f(C)^2 = 0$. By the genus formula, if F' is a general fiber, we have:

$$2g(F') - 2 = F'(K_{S_{min}} + F') \leqslant -2r$$

and thus $r = 1$, $F = C$. Thus S_{min} is ruled by Noether–Enriques.

Suppose to the contrary that S is ruled over a curve D, $g(D) > 0$, i.e. $S_{min} = \mathbb{P}_D(E)$. Then $f(C)$ must lie in a fiber (since there are no non-constant morphisms $\mathbb{P}^1 \to D$). But thus there are finitely many choices for $f(C)$ – these must be the fibers in which we performed the n-blow-ups in f! Thus there are finitely many choices for C (which is a strict transform of $f(C)$). The contradiction means that S_{min} is ruled over $\mathbb{P}^1$.

Existence of S: let P be a pencil of irreducible cubic curves on $\mathbb{P}^2$ (i.e. take cubic equations f_1, f_2 and let $P := \{\lambda_1 f_1 + \lambda_2 f_2\}$) and let $p_1, \dots, p_9$ be the base points of P (i.e. the intersection of $f_1 = 0$ and $f_2 = 0$). Let also S be the blow-up of $\mathbb{P}^2$ at $p_1, \dots, p_9$. Then:

(a) $K_S \sim -3L + \sum_{i=1}^9 E_i$, where L is the strict transform of any line in $\mathbb{P}^2$ and E_i are the exceptional curves at p_i's.

(b) $-K_S \sim \widetilde{C}$ for any $C \in P$. Indeed, $\widetilde{C} \sim \pi^* C - \sum_{i=1}^9 E_i \sim 3L - \sum_{i=1}^9 E_i$.

(c) $-K_S$ is nef. Indeed, since $-K_S \sim \widetilde{C}$, it suffices to check that $(-K_S)^2 \geqslant 0$. This is immediate:

$$(-3L + \sum_{i=1}^9 E_i)^2 = 9 - 9 = 0.$$

(d) If C is an irreducible curve and $C.(-K_S) = 0$ then $C \equiv_{num} -qK_S$ for some $q \in \mathbb{Q}_+$. Indeed, by Hodge index theorem, since $-K_S$ is nef and $C \in \langle -K_S \rangle^\perp$, we have $C^2 \leqslant 0$. If $C^2 < 0$, then by genus formula we would obtain $C^2 = -2$, $g(C) = 0$. But this is impossible:

Lemma: S doesn't contain rational curves with $C^2 = -2$.
Proof: note that C is a strict transform of a plane curve of degree d. Then $C \sim dL - \sum_{i=1}^9 a_i E_i$ for $a_i \geqslant 0$. Then:

$$0 = C.K = -3d + \sum_i a_i$$

$$-2 = C^2 = d^2 - \sum_i a_i^2$$

i.e. $\sum_i a_i^2 = (\frac{1}{3}\sum_i a_i)^2 + 2$. Let $r := \#\{i : a_i \neq 0\}$. Then by Cauchy–Schwarz inequality: $\sum_i a_i^2 \geqslant \frac{1}{r}(\sum_i a_i)^2$ and

$$-2 \leqslant \frac{1}{9}(r-9) \cdot \sum_i a_i^2.$$

If $a_i \in \{0, 1\}$ for all i, then $3d = r$ and $d^2 = r - 2$. Thus $d^2 - 3d + 2 = 0$ and $d \in \{1, 2\}$. Thus C is a transform of a line or a quadric and either three of points $p_1, \dots, p_9$ would have to lie on a line, or six of points $p_1, \dots, p_9$ would have to lie on a quadrics. Contradiction! Now suppose that $a_i \geqslant 2$ for at least one i. Then $\frac{1}{9}(r-9)(\sum_i a_i^2) \leqslant \frac{1}{9}(r-9)(r+3)$, which is less then -2 for $r \neq 8$. In the case $r = 8$, one has to perform easy but tedious analysis. See Friedman, p. 127 for the full proof.

Thus $C^2 = 0$, which implies by Hodge Index Theorem that $C \equiv_{num} -qK_S$ for some $q \in \mathbb{Q}_+$.

Claim: Any divisor $D \in \mathrm{Div}(S)$ with $D^2 = -1$, $K_S.D = -1$ is equivalent to an exceptional curve.

Proof of claim 1: We start by showing that it is equivalent to an effective divisor. By Riemann–Roch:

$$h^0(D) + h^0(K - D) \geqslant \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - K_S.D) = 1 + 0.$$

Note that $K - D$ is not equivalent to an effective divisor, since $(K - D).\widetilde{C} = (K - D).(-K) = -1$. Thus $h^0(D) \geqslant 1$, $|D| \neq \varnothing$ and WLOG $D = \sum_i n_i C_i$ is effective. Note that $1 = (-K).D = \sum_i n_i (-K).C_i$. Since

$(-K).C_i \geqslant 0$ and by previous remarks, WLOG $D = C_1 + \sum_{i>1} n_i C_i$, where $(-K).C_1 = 1$, $C_i \equiv_{num} m_i(-K_S)$ for $m_i \in \mathbb{Q}_+$ and $D \equiv_{num} C_1 + n \cdot (-K_S)$. Thus we have:

$$-1 = D^2 = C_1^2 + 2n \cdot (-K_S).C_1 = C_1^2 + 2n \cdot \widetilde{C}.C_1 = C_1^2 + 2n,$$

but by genus formula $C_1^2 \geqslant -2$ and thus $n = 0$, $C_1^2 = -1$, $g(C_1) = 0$. This shows the claim.

Claim 2: exceptional curves are in bijection with the lattice $\langle [K_S] \rangle^{\perp} / \langle [K_S] \rangle$ (where $[K_S]$ is the numerical class of K_S).

Proof of claim 2: fix an exceptional curve, e.g. E_1. We claim that the bijection is given by:

$$\begin{array}{rcl} \text{exceptional curves} & \leftrightarrow & \langle [K_S] \rangle^{\perp} / \langle [K_S] \rangle \\ C & \mapsto & [C - E_1] \\ D + E_1 + nK & \leftarrow & [D], \end{array}$$

where $n = \frac{1}{2}D^2 - D.E_1$ (note that $2|D^2$ by the genus formula). Indeed, by Claim 1, exceptional curves are in bijection with divisor classes C such that $C^2 = C.K = -1$. Thus if C is such a class then $(C - E_1).K = -1 - (-1) = 0$. The other way around, if $[D] \in \langle [K_S] \rangle^{\perp} / \langle [K_S] \rangle$ then $(D + E_1 + nK).K = E_1.K = -1$ and:

$$(D + E_1 + nK)^2 = D^2 + E_1^2 + n^2 K^2 + 2D.K + 2nE_1.K + 2D.E_1 = D^2 - 1 + 0 + 0 - 2n + 2D.E_1,$$

which equals to -1 iff $n = \frac{1}{2}D^2 - D.E_1$.

End of the proof: $\rho(S) = 10$ and thus $\langle [K_S] \rangle^{\perp} / \langle [K_S] \rangle \cong \mathbb{Z}^8$ Note that $\rho(\mathbb{P}^2) = 1$ and each blow-up increases ρ by one. Thus $\rho(S) = 1 + 9 = 10$. This means that S has infinitely many exceptional curves.

(Note that this does not contradict Hartshorne, AG, Corollary V.5.4. - only finitely many of those curves are contracted by the map $S \to \mathbb{P}^2$; the image of the rest of them are some rational curves)

VI Surfaces with $p_g = 0$, $q \geqslant 1$

VI (1) By the genus formula:

$$0 = 2g_H - 2 = H^2 + H.K.$$

Thus $H.K = -H^2 < 0$ (since H^2 equals the degree of S in $\mathbb{P}^n$, it is positive) and by Corollary VI.18 (2), S must be ruled.

Suppose to the contrary that $q(S) \geqslant 2$. Then we obtain a morphism $\varphi : S \to C$ to a smooth projective curve C of genus $q(S)$ (by composing $S \to S_{min}$ with $S_{min} \to C$ – recall that S_{min} is geometrically ruled). Any morphism $H \to C$ must be constant, since $g(H) \leqslant g(C)$. Thus H is contained in a fiber f of φ. But $f^2 = 0$ (since any two fibers are algebraically equivalent) and on the other hand $f = \sum_i n_i f_i + nH$ for $n > 0$, $n_i \geqslant 0$, implying $f^2 > 0$. Contradiction means that $q(S) \leqslant 1$.

If $q(S) = 1$ then S is ruled over a curve of genus 1, i.e. an elliptic curve.

Suppose finally that $q(S) = 0$. Note that then $\chi(\mathcal{O}_S) = 1 + p_a(S) = 1 + q(S) + p_g(S) = 1$. Moreover, by Kodaira vanishing $h^1(K + H) = h^2(K + H) = 0$. Thus, by Riemann–Roch:

$$h^0(K + H) = \chi(\mathcal{O}(K + H)) = \chi(\mathcal{O}_S) + \frac{1}{2}\left((H + K)^2 - (H + K).K\right) = 1 + \frac{1}{2}(H + K).H = 1 + 0.$$

Thus $|K + H| \neq \varnothing$. Let $D \in |K + H|$ – then $D.H = (K + H).H = 0$ and on the other hand $D.H$ is the degree of D, since H is very ample. Thus $D = 0$ and $K + H \sim 0$. By ex. V.21(2) this is possible iff S is S_d or S_8'.

S_d and S_8' have elliptic sections: note that $K + H \sim 0$ automatically implies that $2g_H - 2 = H.(H + K) = 0$ and $g_H = 1$.

Example of elliptic ruled surface with elliptic sections: ??? ??

VI (2)

Remark: (b) is not true.

Let H be a smooth hyperplane section of S. Consider the exact sequence:

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(H) \to \mathcal{O}_H(H) \to 0 \qquad (*)$$

and note that $\mathcal{O}_H(H) = \mathcal{O}_H(D)$ for an effective $D \in \mathrm{Div}(H)$, $\deg D = H.H$ (one can denote D as $H \cdot H$). By taking the long exact sequence of $(*)$ we obtain:

$$0 \to H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_S(H)) \to H^0(\mathcal{O}_H(D)) \to H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_S(H)) \to H^1(\mathcal{O}_H(D)) \to \ldots (**)$$

Observe that $H^0(\mathcal{O}_S) = k$, $\dim H^1(\mathcal{O}_S) = q$ and $\dim H^0(\mathcal{O}_S(H)) = n+1$ (since the morphism given by the very ample divisor H embedds S into $\mathbb{P}^n$ and the image is not contained in any hyperplane). Note also that by the genus formula:

$$2g_H - 2 = H^2 + H.K$$

and thus: $(2g_H - 2) - \deg D = H.K$.

Finally, note that the degree of S in $\mathbb{P}^n$ is $d = \deg D (= H.H)$ (cf. Hartshorne, exercise V.1.2).

(a) Note that $H.K \geqslant 0$ by Corollary VI.18 (2) and thus $0 \leqslant \deg D \leqslant 2g_H - 2$. Thus, by Clifford theorem and by $(**)$:

$$n+1 = \dim H^0(\mathcal{O}_S(H)) \leqslant H^0(\mathcal{O}_S) + \dim H^0(\mathcal{O}_H(D)) = 1 + \dim H^0(\mathcal{O}_H(D)) \leqslant 1 + (\frac{1}{2}\deg D + 1)$$

which leads to $\deg D \geqslant 2n-2$.

Suppose that equality holds. Then, by Clifford's theorem (cf. Hartshorne, AG, Theorem IV.5.4), we have three cases to consider:

1^o $D = 0$ – this is impossible, since $d = \deg D > 0$.

2^o $D = K_H$ – then $\deg D = 2g_H - 2$ and (by the above formulas) $K_S.H = 0$. Suppose that $E \in |nK_S|$ for $n \geqslant 0$. Then $E.H = nK_S.H = 0$. Since H is very ample and E – effective, this is possible only if $E = 0$. Thus $|nK_S| = \{0\}$ or $|nK_S| = \varnothing$ for all n. The second case implies that S is ruled (Corollary VI.18 (4)), so we are left with $K_S \sim 0$.

3^o D is a degree 2 divisor on the hyperelliptic curve H with $h^0(D) = 2$. Then $2 = \deg D = 2n-2$ and $n = 2$. Thus we have $S \subset \mathbb{P}^2$ and $S = \mathbb{P}^2$, which is false.

(b) This is not true. See Suzuki's solutions for a counterexample.

VI (3) Note firstly that if S is bielliptic, then $12K \equiv 0$ and $h^0(12K) = h^0(\mathcal{O}_S) = 1$.

Suppose now that S is **not** bielliptic. We consider two cases, just as in Theorem VI.13.

1^o (F is not elliptic)

By the proof of Proposition VI.15:

$$P_{12}(S) = \max\{\deg \mathcal{L}_{12} + 1, 0\}, \text{ where } \deg \mathcal{L}_{12} = -24 + \sum_P [12 \cdot (1 - 1/e_P)]$$

and where e_P are ramification indicies of $F \to F/G$. Let r be the number of ramification points and suppose $e_1 \geqslant e_2 \geqslant \ldots$. By Riemann–Hurwitz formula: $\sum_i (1 - 1/e_i) \geqslant 2$. Again, we divide into subcases:

(a) $r \geqslant 5$.

Then, for any ramification point $[12 \cdot (1 - 1/e_P)] \geqslant 6$ and $P_{12}(S) \geqslant 6 + 1 = 7$.

(b) $r = 4$.

If $e_1 \geqslant 3$ then $[12 \cdot (1 - 1/e_1)] \geqslant 8$ and $[12 \cdot (1 - 1/e_1)] \geqslant 6$ for $i = 2, 3, 4$ and thus $P_{12}(S) \geqslant 3$. Suppose now that $e_1 = \ldots = e_4 = 2$. Then, by Riemann–Hurwitz $2 \leqslant 2g(F) - 2 = -2n + 4$ and thus $n = 1$ and $B \to B/G$ is an isomorphism, contradiction.

(c) $r = 3$.

Recall that $\sum_i (1 - 1/e_i) \geqslant 2$, i.e. $1 \geqslant \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3}$, which implies that either $e_1, e_2, e_3 > 3$ or $(e_1, e_2, e_3) \in \{(3,3,3), (2, \geqslant 3, \geqslant 4)\}$. One easily checks that in each of those cases $P_{12}(S) \geqslant 2$.

(d) $r \leqslant 2$ is impossible, since $\sum_i (1 - 1/e_i) \geqslant 2$.

2^o (B is not elliptic)

By the proof of Proposition VI.15:

$$P_{12}(S) = h^0(B/G, D), \text{ where } D = \sum_P [12 \cdot (1 - 1/e_P)]P$$

and where e_P are ramification indicies of $B \to B/G$. Note that since $g(B) \neq g(B/G) = 1$, there must be at least one ramification index $e_{P_0} > 1$, which implies that $[12 \cdot (1 - 1/e_{P_0})] \geqslant 6$ and $\deg D \geqslant 6$. Thus, by Riemann–Roch:

$$P_{12}(S) = h^0(B/G, D) = \deg D \geqslant 6.$$

VI (4) (Errata: one should suppose that S admits a morphism to a non-rational curve?)

Suppose that $p : S \to B$ is a surjective morphism. Then, we can assume that B is normal (by replacing B by its normalization and using the universality property of normalization). Moreover, by Stein factorisation, we can assume that p has connected fibers. Then by Proposition X.10 $0 = \chi_{top}(S) \geqslant \chi_{top}(B) \cdot \chi_{top}(F_\eta)$. Note that $\chi(F_\eta) = 2 - 2g(F_\eta) \leqslant 0$, since if we would have $g(F_\eta) = 0$ then S would be ruled by Noether–Enriques Theorem. By assuption $g(B) \geqslant 1$. Consider the following cases:

1^o $g(F_\eta) = 1$.

In this case we have an equality in the inequality from Proposition X.10, which implies (after analyzing the proof) that the fibers of $p : S \to B$ are smooth, i.e. p is smooth. Moreover, they are of genus 1. Thus, by Proposition VI.8: $S \cong (B \times F)/G$ and by Lemma VI.10 we can assume that G acts both on B and F.

2^o $g(F_\eta) \geqslant 2$.

In this case the inequality of Proposition X.10 yields $g(B) = 1$. We proceed in the same way as in 1^o.

VII Kodaira dimension

VII (1)

Lemma Let R be a graded integral $\mathbb{C}$-algebra with field of fractions K. Suppose that the transcendence degree of R over $\mathbb{C}$ is d. Then there exist algebraically independent (over $\mathbb{C}$) elements $f_1, \ldots, f_d \in R$, which are homogeneous of the same degree.

Proof: Choose any algebraically independent (over $\mathbb{C}$) elements $f_1, \ldots, f_d \in R$. Suppose that $f_1, \ldots, f_m$ are already homogeneous of the same degree. If all the homogeneous components of f_{m+1} were algebraically dependent from $f_1, \ldots, f_{m-1}, f_{m+1}, \ldots f_d$, then f_{m+1} would also be dependent. Thus we can replace f_{m+1} by its homogeneous component in such a way that the transcendence degree of $\mathbb{C}(f_1, \ldots, f_d)$ is still d. Thus, after d steps we can assume that $f_1, \ldots, f_d$ are all homogeneous. By replacing f_i's by suitable powers, we can assume that they are of the same degree. This ends the proof.

By the **Lemma**, we can choose $f_1, \ldots, f_d \in \Gamma(V, \mathcal{O}_V(nK))$, which are algebraically independent. Let $f_{d+1}, \ldots, f_N$ be such that $f_1, \ldots, f_N$ is a basis of $\Gamma(V, \mathcal{O}_V(nK))$. We want to show that $\varphi_{|nK|}(V)$ has dimension at least $d-1$. Note that on $U := \{x : f_1(x) \neq 0\}$, $\varphi_{|nK|}$ is given by $[1 : f_2/f_1 : \ldots : f_m/f_1]$. Note that the coordinates 2 to d are algebraically independent, and thus the image of U has dimension $d-1$. This ends the proof.

VII (2) Let $S = \bigoplus_{n \geqslant 0} H^0(\mathcal{O}_V(nK_V)) = \bigoplus_n S_n$, $T = \bigoplus_{n \geqslant 0} H^0(\mathcal{O}_W(nK_W)) = \bigoplus_n T_n$. By Fact III.22 (i) and (ii):

$$H^0(\mathcal{O}_{V \times W}(nK_{V \times W})) = H^0(\mathcal{O}_V(nK_V)) \otimes_{\mathbb{C}} H^0(\mathcal{O}_W(nK_W)) = S_n \otimes_{\mathbb{C}} T_n,$$

i.e.

$$\bigoplus_{n \geqslant 0} H^0(\mathcal{O}_{V \times W}(nK_{V \times W})) = \bigoplus_n S_n \otimes_{\mathbb{C}} T_n$$

is the **cartesian product of the graded $\mathbb{C}$-algebras** S **and** T, $S \times_{\mathbb{C}} T$ (cf. Hartshorne, Algebraic Geometry, Exercise II.5.11). By the same exercise in Hartshorne:

$$\operatorname{Proj} \bigoplus_{n \geqslant 0} H^0(\mathcal{O}_{V \times W}(nK_{V \times W})) = \operatorname{Proj} S \times_{\mathbb{C}} \operatorname{Proj} T.$$

Thus the dimension of the above scheme is $\dim \operatorname{Proj} S + \dim \operatorname{Proj} T$ and therefore the transcendence degree of $\bigoplus_{n\geqslant 0} H^0(\mathcal{O}_{V\times W}(nK_{V\times W}))$ equals the transcendence degree of S plus the transcendence degree of T, which ends the proof by the previous exercise.

VII (3) We will use the following Lemma:

Lemma (MO80288) Let $\pi : V \to W$ be a generically separable surjective morphism of projective smooth varieties of the same dimension. Then:

$$K_V - \pi^* K_W \geqslant 0.$$

Moreover, if π is étale, $K_V = \pi^* K_W$.

Proof: consider the relative cotangent exact sequence:

$$0 \to \pi^*\Omega_{W/k} \to \Omega_{V/k} \to \Omega_{V/W} \to 0$$

(it is exact on the left, since π is generically separable and $\dim V = \dim W$, cf. [Ravi Vakil, Foundations, Proposition 21.7.2]). By taking determinant, we see that $\pi^*\omega_{W/k} \subset \omega_{V/k}$. This ends the proof of the first part. The second is straightforward, since in that case $\Omega_{V/W} = 0$ by definition of étale morphism.

Note that by projection formula, since K_W is a line bundle, $\pi_*\pi^*K_W = K_W$ and thus

$$H^0(W, nK_W) \cong H^0(W, \pi_* n\pi^* K_W) \cong H^0(V, n\pi^* K_W)$$

and the last space embedds into $H^0(V, nK_V)$ by the Lemma. Thus the canonical ring of W embedds into that of V and $\kappa(W) \leqslant \kappa(V)$. If π is étale then $K_V = \pi^* K_W$ and the canonical rings are equal, which leads to the conclusion.

VIII Surfaces with $\kappa = 0$

VIII (1) (Errata: probably, it was meant to be $P_{12} > 1$?)

Observe that S is non-ruled, as otherwise $p_g = 0$. Note that by assumption $p_a(S) = p_g(S) - q(S) = -1$ and thus $\chi(\mathcal{O}_S) = 0$. Thus, by Theorem X.4, $\kappa(S) < 2$, which implies (by Lemma IX.1 and Proposition VI.2) that $K_S^2 = 0$. Thus by Noether formular, $\chi_{top}(S) = 12\chi(\mathcal{O}_S) = 0$. Therefore by ex. VI.4, $S = (B \times F)/G$, where B is an elliptic curve. Also, we can assume that G acts on B and F by Lemma VI.10. Now, by proof of Theorem VI.13, $2 = q(S) = g(B/G) + g(F/G)$. Note that $g(B/G) \leqslant g(B) = 1$. We consider the following two cases:

1^o $g(B/G) = 1$. In this case $g(F/G) = 1$. This is possible iff we have group monomorphisms $\phi_1 : G \to B$, $\phi_2 : G \to F$ and G acts via translations via ϕ_1, ϕ_2. But then G might be considered as a subgroup of $B \times F$ via $(\phi_1, \phi_2) : G \to B \times F$ and a quotient of an abelian variety by a subgroup is an abelian variety.

2^o $g(B/G) = 0$. In this case $g(F/G) = 2$. We'll show that $P_2 > 1$. By proof of Proposition VI.15:

$$P_{12} = \dim \left(H^0(\omega_B^{\otimes 12}) \otimes H^0(\omega_F^{\otimes 12})\right)^G$$

Note that $H^0(\omega_B^{\otimes 12})$ is G-invariant (one checks that via the explicit description of automorphisms of an elliptic curve). Thus

$$P_{12} = \dim H^0(\omega_F^{\otimes 12})^G = \dim H^0(F/G, \mathcal{L}_{12}), \text{ where } \mathcal{L}_{12} = \omega_{F/G}^{\otimes 12} \otimes \mathcal{O}\left(\sum_{P\in F/G} [12 \cdot (1 - 1/e_P)]\right),$$

and e_P are the ramification indices of $F \to F/G$. Note that:

$$\deg \mathcal{L}_{12} = 12 \cdot (2 \cdot 2 - 2) + \sum_P [12 \cdot (1 - 1/e_P)].$$

Thus $\deg \mathcal{L}_{12} > 2g(F/G) - 2 = 2$ and the proof follows by Riemann–Roch.

VIII (10) Consider the following two cases:

(a) $g = 2k+1$. Let S be the surface given by the equation:

$$w^2 = f(x,y), \quad (x,y) \in \mathbb{P}^1 \times \mathbb{P}^1$$

(where f is of bidegree 4, e.g. ???) inside of ?? $\mathbb{WP}(2,1,1)$??. Note that S is smooth ??.

Consider the map $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$, $\pi(x,y,w) = (x,y)$. This is a double cover branched along $C_1 : f(x,y) = 0$. Thus by Riemann–Hurwitz formula:

$$K_S = \pi^* K_{\mathbb{P}^1 \times \mathbb{P}^1} + R_{S/\mathbb{P}^1 \times \mathbb{P}^1} = \pi^*(-2H_1 - 2H_2) + C_2$$

(where $C_2 : w = f(x,y) = 0 \subset S$). But $\pi^*(C_1) = e(C_1) \cdot C_2 = 2 \cdot C_2$ and on the other hand $\pi^*(C_1) = \pi^*(-4H_1 - 4H_2)$. Therefore:

$$K_S \sim 0$$

and S is a K3 surface.

Let C be the preimage in S of any smooth curve in $|H_1 + kH_2|$. Then $\varphi_{|C|}$ is a composition of:

$$S \xrightarrow{\pi} \mathbb{P}^1 \times \mathbb{P}^1 \overset{\varphi_{|H_1+kH_2|}}{\hookrightarrow} \mathbb{P}^{2k+1}$$

– this ends the proof in this case.

(b) Let $C' : f(x,y,z) = 0$ be any nodal sextic in $\mathbb{P}^2$, e.g. ??, with node in P_0. Let S' be given by the equation:

$$w^2 = f(x,y,z)$$

in ?? the weighted projective space $\mathbb{WP}(3,1,1)$. Let also $\pi' : S' \to \mathbb{P}^2$, $(x,y,z,w) \mapsto (x,y,z)$ – it is a double cover, branched in C'. Consider now the blow-ups in P_0 and $\pi'^{-1}(P_0)$:

$$\begin{array}{ccc} S := Bl_{\pi^{-1}(P_0)}(S') & \xrightarrow{\pi} & \mathbb{F}_1 := Bl_{P_0}(\mathbb{P}^2) \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\pi'} & \mathbb{P}^2 \end{array}$$

VIII (12) Consider the universal coefficient theorem for cohomology:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{i-1}(X,\mathbb{Z}),\mathbb{Z}) \to H^i(X;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_i(X;\mathbb{Z}),\mathbb{Z}) \to 0.$$

Note that for any finitely generated abelian group M:

- $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ is torsion-free,
- $\mathrm{Ext}_{\mathbb{Z}}(M,\mathbb{Z}) \cong M_{tors}$, since $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/n,\mathbb{Z}) \cong \mathbb{Z}/n$, $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \cong 0$.

Therefore, $H^i(X;\mathbb{Z})_{tors} \cong H_{i-1}(X,\mathbb{Z})_{tors}$.

Let S be a K3 surface. Then $b_1(S) = 2q(S) = 0$ and thus $H_1(S,\mathbb{Z})$ is finite. Note that $H_1(S,\mathbb{Z}) = \pi_1(S,s)^{ab}$. Let $\pi : \widetilde{S} \to S$ be an étale cover of S of degree n. Then by ex. VII (3), $\kappa(\widetilde{S}) = 0$ and thus $p_g(\widetilde{S}) \leqslant 1$. On the other hand:

$$\chi(\mathcal{O}_{\widetilde{S}}) = n \cdot \chi(\mathcal{O}_S) = n \cdot (1 - q(S) + p_q(S)) = 2n$$

and, since $\chi(\mathcal{O}_{\widetilde{S}}) = 1 - q(\widetilde{S}) + p_q(\widetilde{S})$, $p_g(\widetilde{S}) \geqslant 2n-1$. Thus $1 \geqslant 2n-1$ and $n = 1$, i.e. π is an isomorphism. Thus S has no non-trivial étale covers. Therefore $H_1(S,\mathbb{Z}) = 0$ (since every finite topological cover of S is an algebraic surface, which is an étale cover of S) and $H^2(S,\mathbb{Z})_{tors} = H_1(S,\mathbb{Z}) = 0$.

Let S be now an Enriques surface with a double cover $\pi : \widetilde{S} \to S$, where $\widetilde{S}$ is a K3 surface. Then $\pi_1(S,s)/\pi_1(\widetilde{S},\widetilde{s}) \cong \mathbb{Z}/2$ and in particular $H_1(S,\mathbb{Z}) \cong H_1(S,\mathbb{Z})/H_1(\widetilde{S},\mathbb{Z}) \cong \mathbb{Z}/2$. Therefore $H^2(S,\mathbb{Z})_{tors} = H_1(S,\mathbb{Z}) = \mathbb{Z}/2$. Finally, note that $[K]$ (the image of K under $\mathrm{Pic}\, S \to H^2(S,\mathbb{Z})$) is non-zero:

- the kernel of $\mathrm{Pic}\, S \to H^2(S,\mathbb{Z})$ is the complex torus of dimension $q(S) = 0$, i.e. it is trivial,
- $K \nsim 0$, since $p_g(S) = 0 \neq 1$

and $2[K] = [2K] = 0$. Thus $H^2(S,\mathbb{Z})_{tors} = \langle [K] \rangle \cong \mathbb{Z}/2$. This ends the proof.

IX Surfaces with $\kappa = 1$

IX (1)

P **is surjective:** note that $B \subset \operatorname{im} P$ and thus $Jac(B) \subset \operatorname{im} P$, since B generates $Jac(B)$.

$q(S) \in \{g(B), g(B)+1\}$**:** by low degree terms exact sequence for Leray spectral sequence:

$$0 \to H^1(B, \mathcal{O}_B) \to H^1(S, \mathcal{O}_S) \to H^0(B, R^1 p_* \mathcal{O}_S) \to 0$$

(note that $p_* \mathcal{O}_S = \mathcal{O}_B$, since ????). Hence:

$$q(S) = \dim H^1(\mathcal{O}_S) = \dim H^1(\mathcal{O}_B) + \dim H^0(R^1 p_* \mathcal{O}_S) = g(B) + \dim H^0(R^1 p_* \mathcal{O}_S).$$

Consider now $\mathcal{L} := R^1 p_* \mathcal{O}_S$. Note that for all fibers, singular or not, $\dim H^1(F_b, \mathcal{O}_{F_b}) = 1$ (this follows by classification of singular fibers?). Thus $\mathcal{L}$ is a line bundle by Grauert theorem ([Hartshorne, AG, Corollary 12.9])

Moreover, the degree of $\mathcal{L}$ equals $-\chi(\mathcal{O}_S)$. Indeed, by taking the Euler characteristic of the Leray spectral sequence:

$$\begin{aligned} \chi(\mathcal{O}_S) &= \sum_{p,q} (-1)^{p+q} \chi(E_2^{pq}) = \dim H^0(\mathcal{L}) - \dim H^1(\mathcal{L}) + \dim H^1(\mathcal{O}_B) - \dim H^0(\mathcal{O}_B) \\ \chi(\mathcal{O}_S) &= (\deg \mathcal{L} + 1 - g(B)) + (g(B) - 1). \end{aligned}$$

Therefore, by Castelnuovo inequality:

$$\deg \mathcal{L} = -\chi(\mathcal{O}_S) \leqslant 0$$

and we have two possibilities:

- $\mathcal{L} \cong \mathcal{O}_B$ – then $\dim H^0(\mathcal{L}) = 1$ and $q(S) = g(B) + 1$,
- $\mathcal{L} \not\cong \mathcal{O}_B$ – then $H^0(\mathcal{L}) = 0$ and $q(S) = g(B)$.

Kernel of P**:** suppose that $q(S) = g(B) + 1$. Then the kernel of P is one-dimensional, and thus it is an elliptic curve E. Fix an embedding $\beta : B \to Jac(B)$. Let $b \in B$ and suppose that F_b is smooth. Then the fiber of $\beta(b)$ via $Alb(S) \to Jac(B)$ is a translate of E. Thus we obtain a morphism $F_b \to E$ – this means that F_b and E are isogeneous.

Sources: Friedman, Algebraic Surfaces and Holomorphic Vector Bundles; Dürr, Fundamental groups of elliptic fibrations and theinvariance of the plurigenera for surfaces with odd first Betti number.

IX (6)

Step I: WLOG D is effective.

By Riemann–Roch, $h^0(D) + h^0(-D) \geqslant 2$. Thus, (if there exists at least one smooth rational curve), obviously $h^0(D) \geqslant 2$. Therefore we can WLOG assume that D is effective.

Step II: D is nef.

Firstly, note that if D is effective and $D.C \geqslant 0$ for all rational curves then D is nef, i.e. $D.E \geqslant 0$ for every effective divisor E. Indeed, it suffices to check this when E is an irreducible curve. But then if $g(E) \geqslant 1$, then by genus formula $E^2 \geqslant 0$ and thus if $D = nE + D'$ for $n \geqslant 0$, D' not containing E. Thus $D.E = nE^2 + D'.E \geqslant 0$.

Step III: $|D|$ has no fixed part.

Let Z, M be the fixed and mobile part of D. Note that $0 = D^2 = D.Z + D.M$. But $D.Z, D.M \geqslant 0$ (since Z, M are effective and D is nef). Thus $D.Z = D.M = 0$. But $0 = D.M = M^2 + Z.M$, and since $M^2, Z.M \geqslant 0$ (as M is mobile), $M^2 = Z.M = 0$. But $0 = D^2 = M^2 + Z^2 + 2Z.M = 2Z.M$ and thus $Z^2 = 0$. Assume to the contrary that $Z \neq 0$. By Riemann–Roch, $h^0(Z) + h^0(-Z) \geqslant 2$, and thus (since $Z > 0$) $h^0(Z) \geqslant 2$. But Z is the fixed part of $|D|$, and thus $h^0(Z) \leqslant 1$! Contradiction proves that $Z = 0$.

Step IV: $|D|$ is base point free.

$|D|$ has no fixed part, and thus the number of its fixed points is $\leqslant D^2 = 0$.

Step V: $D \sim kE$ for an elliptic curve E and $k \geqslant 1$.

Consider now the morphism $\phi : S \to \mathbb{P}^N$, defined by D. Note that since $D^2 = 0$, its image must be a curve (if its image was a surface, we would obtain a contradiction by Hodge index theorem, cf. Corollary VIII.5). Let $S \to C \to C' \subset \mathbb{P}^N$ be the Stein factorisation of ϕ, where $C \to C'$ is of degree $k \geqslant 0$. Let E be the generic fiber of $S \to C$. Then E is smooth, $E^2 = 0$ and by the genus formula $g(E) = \frac{1}{2}E^2 + 1 = 1$. Thus $D \sim kE$ and the proof follows.

Step VI: $D^2 = 0$, $D \nsim 0 \Rightarrow \quad S$ is elliptic.

Method I:

Lemma: ("Weyl chambers") Let V be an Euclidean space with an indefinite bilinear form $\Phi(X, Y)$ of signature $(1, \dim V - 1)$. Let $T \subset V$ be a finite subset and let:

$$\mathcal{C} := \{x \in V : \Phi(x, t) \geqslant 0 \, \forall_{t \in T}\}.$$

Suppose that $\mathcal{C} \neq \varnothing$. Let $s_t(x) := x + \Phi(x, t)t$ (reflection around $\Phi(x, t) = 0$). Then for any $x \in V$, there exists $s \in \langle s_t : t \in T \rangle$ such that $s(x) \in C$.
Proof: Note that $\mathcal{C}$ is a cone in V. Thus it is given by finitely many inequalities, in particular we may assume that T is finite. The hyperplanes $(\Phi(x, t) = 0)_{t \in T}$ divide V into finitely many chambers. By using the reflections, we can move x from one chamber to any other, in particular to $\mathcal{C}$. (It is a standard proof in the theory of root systems, cf. e.g. Kirillov – Introduction to Lie Groups and Lie Algebras, Lemma 7.26).

Let $V := NS(S) \otimes_{\mathbb{Z}} \mathbb{R}$, $T = \{[C] : C \text{ is a rational curve on } S\}$, $\Phi([D_1], [D_2]) = D_1.D_2$. Note that then:

- WLOG T is finite. (since nef cone is a cone, it can be given by finitely many inequalities – why??)
- $\mathcal{C} \neq \varnothing$, since the class of any ample divisor belongs to $\mathcal{C}$.

Then for some $w \in \langle w_{[C]} : [C] \in T \rangle$, $w(D).C \geqslant 0$ for all $C \in T$. Moreover, if $E^2 = 0$ and $[C] \in T$ then:

$$w_{[C]}(E)^2 = (E + (E.C)C, E + (E.C)C) = E^2 + 2 \cdot (E.C)^2 + (E.C) \cdot C^2 = 0 + 2 \cdot (E.C)^2 + (E.C) \cdot (-2) = 0.$$

Thus $w(D)^2 = 0$ and the proof follows by earlier steps.

Method II: suppose that C is a rational curve, such that $D.C < 0$. Let $D' := w_C(D)$. Then $D'^2 = 0$, $D'.C = D.C - 2D.C = -D.C$. Moreover, one shows that if $\dim |D| \geqslant 1$ then $\dim |D'| \geqslant 1$. Finally, note that $0 < H.D' = H.D + (C.D)C.H < H.D$, so this procedure may be performed only finitely many times.

IX (7) We start by computing the Picard number of S, i.e. $\rho(S) := \text{rank}_{\mathbb{Z}} NS(S)$. Note that $p_a(S) = p_g(S) - q(S) = 0$ and thus $\chi(\mathcal{O}_S) = 1$. But $\chi(\mathcal{O}_S) = h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S) = 1 - q(S) + h^2(\mathcal{O}_S) = 1 + h^2(\mathcal{O}_S)$. By the exponential sequence we obtain:

$$0 \to H^1(S^{an}, \mathbb{Z}) \to H^1(S, \mathcal{O}_S) \to \text{Pic}(S) \to H^2(S^{an}, \mathbb{Z}) \to H^2(S, \mathcal{O}_S) = 0.$$

Therefore:

$$NS(S) := \text{im}(\text{Pic}(S) \to H^2(S^{an}, \mathbb{Z})) = H^2(S^{an}, \mathbb{Z})$$

and $\rho(S) = b_2(S)$. By Noether formula we have:

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi_{top}(S)) \Rightarrow \chi_{top}(S) = 12\chi(\mathcal{O}_S) - 0 = 12.$$

On the other hand, $\chi_{top}(S) = 2 - 2b_1(S) + b_2(S) = 2 - 4q(S) + b_2(S)$ and thus $b_2(S) = 10$. We use now the following fact:

Fact: Let V be a $\mathbb{Q}$-vector space of dimension $\geqslant 5$. Every indefinite quadratic form on V admits a non-trivial zero.
(It is an easy corollary of Hasse principle for quadratic forms, cf. [Serre, A Course in Arithmetic, p. 38])

Consider now the quadratic form $D \mapsto D.D$ on $NS(S) \otimes \mathbb{Q}$. It is indefinite (of signature $(1, -1, -1, -1 \dots, -1)$ by Hodge index theorem). By the fact it admits a non-trivial zero, which yields us a divisor $D \in \mathrm{Div}(S)$, $D^2 = 0$, $D \not\equiv 0$. Let $\pi : \widetilde{S} \to S$ be the associated covering by a K3-surface. Then $\pi^* D \not\sim 0$ (since $\pi_* \pi^* D = 2D \not\sim 0$) and $(\pi^* D)^2 = D^2 = 0$. Thus $\widetilde{S}$ is elliptic by the previous exercise. Let $\widetilde{S} \to \mathbb{P}^1$ be the elliptic fibration and suppose that it is given by a linear system P. Consider now the linear system $\pi_* P$. Note that its generic member is covered by an elliptic curve from P, so it is also an elliptic curve by Riemann–Hurwitz formula. Also, it is base point free. Indeed, if $x \in S$ would be a base point and $\pi^{-1}(x) = \{x_1, x_2\}$, then every member of P would pass through x_1 or x_2. But then

$$P = \{D \in P : \ D \text{ passes through } x_1 \} \cup \{D \in P : \ D \text{ passes through } x_2 \}$$

– by the irreducibility of projective space, P would be equal to one of those sets, and would have a base point. Thus $\pi_* P$ gives a morphism into projective space, whose generic fiber is an elliptic curve.

X Surfaces of general type

X (1) (Stolen from Suzuki)

Note that there exists a composition of blow-ups $\varepsilon : \widetilde{S} \to S$ such that ϕ_K lifts to a morphism $\phi : \widetilde{S} \to S'$. Let $K_S = Z + M$ be the fixed and mobile part of K_S. Then by the above assumption, the divisor $M' = \varepsilon^* M - \sum_i a_i E_i$ (where $a_i \geqslant 1$ and E_i are exceptional curves on $\widetilde{S}$) is base point free and defines $\phi : \widetilde{S} \to S'$. We consider two cases:

1^o $\phi_K(S)$ is a surface S'.

Note that S' is a surface of degree $(M')^2$ (since $M' = \phi^* H$ for a hyperplane section H) in $|K|^* = |(M')^*|$ and that:

$$(M')^2 = M^2 - \sum_i a_i^2 \leqslant M^2$$

But $h^0(K) = p_g$ and thus $\dim |K|^* = \dim |M'|^* = p_g - 1$. Therefore by ex. VI.2 (a) $M'^2 \geqslant 2(p_g - 1) - 2 = 2p_g - 4$. On the other hand:

$$K^2 = Z^2 + M^2 + 2Z.M = K.Z + Z.M + M^2 \geqslant M^2$$

(since S is of general type, K_S is nef by Corollary VI.18 (2) – thus $K.Z \geqslant 0$. Moreover, $Z.M \geqslant 0$, since M is mobile and may be assumed to have no common components with Z). This ends the proof in this case.

2^o $\phi_K(S)$ is a curve C.

Idea: ϕ_K cannot be a morphism – otherwise $K^2 = (n \cdot \text{fiber}) = 0$. Also we can estimate n (the degree of finite morphism in Stein factorisation).

Since $K^2 > 0$, we can WLOG assume that $p_g \geqslant 3$. Let $\widetilde{S} \to \widetilde{C} \to C$ be the Stein factorisation, where $\widetilde{C} \to C$ is a finite morphism of degree n.

Step I: $n \geqslant p_g - 1$.

Observe that $M' = F_1 + \ldots + F_n$, where F_i are the connected components of the fiber of $\widetilde{S} \to C$ (if F is a fiber of $\widetilde{S} \to \widetilde{C}$ then $F \equiv_{alg} F_i$). Consider now the exact sequence:

$$0 \to \mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{S}}(M) \to \mathcal{O}_M(M) = \bigoplus_i \mathcal{O}_{F_i}(M \cdot F_i) = \bigoplus_i \mathcal{O}_{F_i}(F_i \cdot F_i)$$

and the associated long exact sequence:

$$0 \to H^0(\mathcal{O}_{\widetilde{S}}) \to H^0(\mathcal{O}_{\widetilde{S}}(M)) \to \bigoplus_i H^0(\mathcal{O}_{F_i}(M \cdot F_i)) \to \ldots,$$

which yields:

$$\sum_i \dim H^0(\mathcal{O}_{F_i}(M \cdot F_i)) \geqslant \dim H^0(\mathcal{O}_{\widetilde{S}}(M)) - \dim H^0(\mathcal{O}_{\widetilde{S}}) = p_g - 1.$$

On the other hand,

$$\dim H^0(\mathcal{O}_{F_i}(M \cdot F_i)) = \dim H^0(\mathcal{O}_{F_i}) = 1$$

(since $F_i \equiv_{num} F$, $F_i^2 = 0$) and thus $n \geqslant p_g - 1$.

Step II: note that ε is proper and thus we have a pushforward on divisors. Let $F_1 := \varepsilon_* F$. Then $M \equiv_{alg} nF_1$. And thus (since K is nef):

$$K^2 = K.Z + K.M \geqslant K.M = n \cdot (K.F_1).$$

Thus it suffices to show that $K.F_1 \geqslant 2$ – then we will have:

$$K^2 = n \cdot (K.F_1) \geqslant 2n \geqslant 2 \cdot (p_g - 1).$$

Step III: $K.F_1 \geqslant 2$.

Suppose to the contrary that $K.F_1 \in \{0, 1\}$. If $K.F_1 = 1$ then:

$$1 = M.F_1 + Z.F_1 = nF_1^2 + Z.F_1.$$

But $Z.F_1 = \frac{1}{n} Z.M \geqslant 0$ and $M.F_1 = \frac{1}{n} M^2 \geqslant 0$ and thus we have two possibilities:

1^o $F_1^2 = n = 1$, $Z.F_1 = 0$.

In this case $1 \geqslant p_g - 1$, i.e. $p_g \leqslant 2$ and we are done by previous remark.

2^o $F_1^2 = 0$, $Z.F_1 = 1$.

By genus formula: $2 | F_1^2 + Z.F_1 = 1$, which yields contradiction.

If $K.F_1 = 0$ then $Z.F_1 + M.F_1 = 0$ and thus $F_1^2 = 0$. But, since $K_S^2 > 0$, by Hodge index theorem, $F_1 \equiv_{num} aK$ for $a \in \mathbb{Q}$, and thus $0 = F_1^2 = aK^2$ implies $a = 0$. But $F_1 \equiv_{num} 0$ is impossible, since F_1 is an irrducible curve! (e.g. if H is very ample then $H.F_1 > 0$). This ends the proof.

X (2) Suppose that $S' \to S$ is an étale cover of S of degree n. Then $\chi(\mathcal{O}_{S'}) = n\chi(\mathcal{O}_S) \geqslant n$, i.e. $1 - q(S') + p_g(S') \geqslant n$, which implies $p_g(S') \geqslant n - 1$. Then by Noether inequality (Ex. X(1)) $K_{S'}^2 = nK_S^2 = n \geqslant 2p_g(S') - 4 = 2(n-1) - 4$, i.e. $n \leqslant 6$. This implies that S has only finitely many étale covers (why???) and thus $\pi_1(S)^{ab} \cong H_1(S, \mathbb{Z})$ is finite, i.e. $0 = b_1(S) = 2q(S)$. This shows also that $\# H_1(X, \mathbb{Z}) \leqslant 6$.

?????

X (3) **Erratum:** in $\mathbb{P}^6$.

Let $[a_{ij}]_{1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 7} \in M_{4,7}(\mathbb{C})$ be any matrix of rank 9 and define:

$$Q_i(X_1, \ldots, X_7) := \sum_{j=1}^{7} a_{ij} X_j^2, \quad S' := Q_1 \cap \ldots \cap Q_4.$$

Let also $G = (\mathbb{Z}/2)^3$ act on $\mathbb{P}^8$ via:

$$\begin{aligned}
(1,0,0) \cdot [X_1 : X_2 : \ldots : X_7] &= [-X_1 : -X_2 : -X_3 : -X_4 : X_5 : X_6 : X_7] \\
(0,1,0) \cdot [X_1 : X_2 : \ldots : X_7] &= [-X_1 : -X_2 : X_3 : X_4 : -X_5 : -X_6 : X_7] \\
(0,0,1) \cdot [X_1 : X_2 : \ldots : X_7] &= [X_1 : -X_2 : -X_3 : X_4 : X_5 : -X_6 : -X_7]
\end{aligned}$$

One easily checks that for every $g \in G$, $g \neq 0$:

$$g \cdot [X_1 : X_2 : \ldots : X_7] = [\varepsilon_1 X_1 : \varepsilon_2 X_2 : \ldots : \varepsilon_7 X_7]$$

where $\varepsilon_i \in \{\pm 1\}$ and among ε_i there are three 1's and four -1's, or three -1's and four 1's. We will show that if $P \in S'$, $g \in G$, $g \neq e$ then $g \cdot P \neq P$. Suppose the opposite. The equality $g \cdot P = P$ implies that at least three numbers from $\{X_1, \ldots, X_9\}$ are zero. Indeed, WLOG we can check it for $g = (1,0,0)$. If $[X_1 : X_2 : \ldots : X_7] = [-X_1 : -X_2 : -X_3 : -X_4 : X_5 : X_6 : X_7]$ then either $X_5 = X_6 = X_7 = 0$ or $(X_1, X_2, \ldots, X_7) = (-X_1, -X_2, -X_3, -X_4, X_5, X_6, X_7)$ and thus $X_1 = X_2 = X_3 = X_4 = 0$.

Thus the squares of the non-zero coordinates of P satisfy the system of 4 linear equations $Q_i(P) = 0$ for $i = 1, \ldots, 4$. Since this system has 4 equations, 4 variables and rank 4, all the solutions are zero. This ends the proof of the fact that G acts on S' freely.

Note that $\mathcal{O}_S(K_S) = \mathcal{O}_S(4 \cdot 2 - 7) = \mathcal{O}_S(H)$ and thus $K_S^2 = H^2$ = degree of S' in $\mathbb{P}^6 = 2^4 = 16$. Since $S' \to S$ is étale, $K_{S'} = \pi^* K_S$ and (by Prop. I.8 (ii)) $K_S^2 = \frac{1}{8} K_{S'}^2 = 2$. To compute $q(S)$, note that $q(S') = \dim H^1(\mathcal{O}_{S'}) = 0$ (since S' is a complete intersection) and thus:

$$q(S) = \dim H^0(\Omega_S) = \dim H^0(\Omega_{S'})^G = \dim H^1(\mathcal{O}_{S'})^G = 0.$$

Finally, since $\chi(\mathcal{O}_{S'}) = 25 \cdot \chi(\mathcal{O}_S)$, we compute that $p_g(S) = 0$. ??????

X (4) Let $\phi = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathrm{Aut}((\mathbb{Z}/5)^2)$. Then the action of $g = (x, y) \in (\mathbb{Z}/5)^2$ on $C \times C$ is as follows:

$$g \cdot ([X_1 : Y_1 : Z_1], [X_2 : Y_2 : Z_2]) = ([\zeta^x \cdot X_1 : \zeta^y \cdot Y_1 : Z_1], [\zeta^{x+2y} \cdot X_2 : \zeta^{3x+4y} \cdot Y_2 : Z_2]).$$

Suppose that $g \cdot (P_1, P_2) = (P_1, P_2)$ with $g \neq (0, 0)$. Consider the following possibilities:

1^o $Z_1, Z_2 \neq 0$.

Then $X_1 = \zeta^x \cdot X_1$ and thus $X_1 = 0$ or $x = 0$. Analogously, $Y_1 = 0$ or $y = 0$, $X_2 = 0$ or $x + 2y = 0$, $Y_2 = 0$ or $3x + 4y = 0$. Note that $(X_1, Y_1), (X_2, Y_2) \neq (0, 0)$. Thus we have two possibilities:

1^o A) $x = 0$. Then $2y = 0$ or $4y = 0$ – both cases lead to $y = 0$, which is a contradiction.

1^o B) $y = 0$. Then $x = 0$ or $3x = 0$ – both cases lead to $x = 0$, which is a contradiction.

2^o $Z_1 \neq 0$, $Z_2 = 0$.

Then $X_1 = 0$ or $x = 0$ and $Y_1 = 0$ or $y = 0$. Moreover, $X_2 = -Y_2 \neq 0$ and $[X_2 : Y_2] = [\zeta^{x+2y} \cdot X_2 : \zeta^{3x+4y} \cdot Y_2] = [\zeta^{(x+2y)-(3x+4y)} \cdot X_2 : Y_2]$ and thus $(x + 2y) - (3x + 4y) = 0$, i.e. $-2x - 2y = 0$. Thus, if one of the numbers x, y is zero, the second is also. Contradiction!

3^o $Z_1 = 0$, $Z_2 \neq 0$.

Then, analogously as in 2^o, $x - y = 0$ and, analogously as in 1^o, $X_2 = 0$ or $x + 2y = 0$, $Y_2 = 0$ or $3x + 4y = 0$. Thus $x + 2x = 0$ or $3x + 4x = 0$ – in both cases $x = y = 0$ – contradiction!

4^o $Z_1 = 0$, $Z_2 = 0$.

Then, analogously as in 2^o, $x - y = 0$ and $(x + 2y) - (3x + 4y) = 0$, which leads to $x = y = 0$. Contradiction!

Thus the action of G on $C \times C$ is free, the quotient $(C \times C)/G$ is a smooth surface and $C \times C \to (C \times C)/G$ is étale of degree $\#G = 25$.

By degree–genus formula $g(C) = 6$. Note that $K_{C \times C} = pr_1^* K_C + pr_2^* K_C$ and thus

$$K_{C \times C}^2 = 2(\deg K_C)^2 = 2 \cdot (2 \cdot (g(C) - 1))^2 = 200.$$

Since $\pi : C \times C \to (C \times C)/G$ is étale, $K_{C \times C} = \pi^* K_{(C \times C)/G}$ and (by Prop. I.8 (ii)) $K_{(C \times C)/G}^2 = \frac{1}{25} \cdot K_{C \times C}^2 = 8$.

Now, analogously as in the proof of Theorem VI.13 and using example VI.12 (a):

$$H^0(\Omega^1_{(C \times C)/G}) = (H^0(\Omega_C)^{\oplus 2})^G, \qquad H^0(\Omega^2_{(C \times C)/G}) = (H^0(\Omega_C)^{\otimes 2})^G.$$

It is a standard fact that

$$H^0(\Omega_C) = \left\{ \frac{x^{i-1}\, dx}{y^j} : (i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) \right\}$$

(where $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$). One checks easily that $(H^0(\Omega_C)^{\oplus 2})^G = (H^0(\Omega_C)^{\otimes 2})^G = 0$ and thus $q = p_g = 0$.

Other examples: ???

X (5) By X (1): $K^2 = 1 \geqslant 2p_g - 4$ and thus $p_g \leqslant 2$. Suppose that $S' \to S$ is an étale cover of S of degree n. Then $\chi(\mathcal{O}_{S'}) = n\chi(\mathcal{O}_S) \geqslant n$, i.e. $1 - q(S') + p_g(S') \geqslant n$, which implies $p_g(S') \geqslant n - 1$. Then by Noether inequality (Ex. X(1)) $K_{S'}^2 = nK_S^2 = n \geqslant 2p_g(S') - 4 = 2(n - 1) - 4$, i.e. $n \leqslant 6$. This implies that S has only finitely many étale covers (why???) and thus $\pi_1(S)^{ab} \cong H_1(S, \mathbb{Z})$ is finite, i.e. $0 = b_1(S) = 2q(S)$. This shows that $q(S) = 0$.

X (6) Suppose to the contrary that image of ϕ_{2K} is a curve C. Let $2K = Z + M$ be the decomposition into fixed and movable part. There exists a composition of blow-ups $\epsilon : \widehat{S} \to S$ such that ϕ_{2K} lifts to a morphism $\phi : \widehat{S} \to C$. In other words, the system $|\widehat{M}|$ has no base points, where $\widehat{M} := \epsilon^* M - \sum_i a_i E_i$, $a_i \geqslant 0$, E_i – exceptional curves.

Let $\widehat{S} \to B \to C$ be Stein factorisation, where $B \to C$ is of degree n and B is smooth. Then $\widehat{M} = \sum_i F_i$, where F_i are fibers of $\widehat{S} \to C$ and $F_i \equiv_{alg} F$ (where F is a generic fiber of $\widehat{S} \to B$). Note that $g(F_i) \geqslant 2$ (otherwise S would be elliptic or ruled). We start by computing n.

By taking the long exact sequence of

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(M) \to \bigoplus_i \mathcal{O}_{F_i} \to 0$$

(?????) and noting that $h^0(M) = h^0(2K) =$ (by Riemann–Roch) $= p_g(S) + 1$, we see that $n = p_g(S)$.

Note that by adjuntion formula, $|K + F|$ induces canonical linear system on F. But canonical system on any smooth curve of genus $\geqslant 2$ is very ample – thus the map defined by $|K + F|$ gives an embedding of F into projective space. We will prove that $K + F \leqslant 2K$. Then it will follow that the map defined by $|K + F|$ factors via the map defined by $|2K|$:

$$\begin{array}{ccc} X & \dashrightarrow & \|2K\|^* \\ & \searrow & \downarrow \\ & & \|K+F\|^* \end{array}$$

This leads to a contradiction, since F is contracted by ϕ_{2K} and $\phi_{K+F}|_F$ is an embedding. Consider two cases:

(a) Suppose that $p_g(S) \geqslant 2$. To show $K + F \leqslant 2K$, it suffices to show that $K + F \leqslant 2K$. But $nF \leqslant 2K$, and thus $K + F \leqslant (1 + \frac{2}{n})K \leqslant 2K$.

(b) Suppose that $p_g(S) = 1$. Then $2K = Z + F$. Thus $F \leqslant K$ (if F is contained in the divisor $2K$, then also in K). Therefore $K + F \leqslant 2K$.

This ends the proof.
TODO: check??? F and $F_i \Rightarrow$??