Szamuely – Galois groups and fundamental groups
Zadania

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1 Galois Theory of Fields

1.2 Let $G = \lim_{\leftarrow} G_i$ be an inverse limit of an inverse system of finite groups. Let $G_i^{(p)}$ denote the corresponding p-Sylow subgroup. Define $G^{(p)} := \lim_{\leftarrow} G_i^{(p)}$. Clearly, it is a pro-$p$-subgroup of G. By the following lemma, any quotient of ?????

> **Lemma** If $H = \lim_{\leftarrow} H_i$, where H_i are finite groups of order non-divisible by p, then for any quotient N of H one has $H_i \twoheadrightarrow N$ for some i.
> **Proof:** let $p_i : H \twoheadrightarrow H_i$ be the canonical projection. Suppose that $p : H \twoheadrightarrow N$, where N is a finite group. The $\ker p$ is an open neighbourhood of identity and thus it contains $\ker p_i$ for some i. Thus $H_i \cong H/\ker p_i \twoheadrightarrow H/\ker p \cong N$. This ends the proof.

1.3 Let $K := k^{sep}$. The given conditions on $k^{(p)}$ are equivalent to $\mathrm{Gal}(K/k^{(p)})$ being the pro-p-Sylow-subgroup of $\mathrm{Gal}(K/k)$. Thus we have to show that if $G^{(p)}$ is the pro-p-Sylow-subgroup of $G := \mathrm{Gal}(K/k)$ then it is closed. Let $K = lim_{\rightarrow} K_n$, where K_n/k are finite Galois and $K_n \subset K_{n+1}$. Then $G^{(p)} = \lim_{\leftarrow} \mathrm{Gal}(K/K_n)^{(p)}$. Let $H_n \subset G$ be the inverse image of $\mathrm{Gal}(K/K_n)^{(p)}$ under

$$\mathrm{Gal}(K/k) \twoheadrightarrow \mathrm{Gal}(K/K_n).$$

Then H_n is closed in G. But (since $H_n \supset H_{n+1}$) $G^{(p)} = \lim_{\leftarrow} H_n = \bigcap_n H_n$ is closed as an intersection of closed subsets.

This extension doesn't have to be unique, since the pro-p-Sylow subgroup is determined only up to conjugation.

1.4 (a) Denote the compositum of all quadratic extensions of $\mathbb{Q}$ by $\mathbb{Q}^{(2)}$. Let $(p_i)_i$ be the sequence of all primes and $\mathbb{Q}_n := \mathbb{Q}(\sqrt{-1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$. Then:

- $\mathbb{Q}^{(2)} = \lim_{\rightarrow} \mathbb{Q}_n$,
- $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) = (\mathbb{Z}/2)^{n+1}$.

Thus $\mathrm{Gal}(\mathbb{Q}^{(2)}/\mathbb{Q}) = \prod_{i=1}^{\infty} \mathbb{Z}/2$. This group clearly has $2^{\aleph_0}$ elements and $2^{\aleph_0}$ subgroups of index 2 (...???).

(b) Note that open subgroups of index 2 of $\mathrm{Gal}(\mathbb{Q}^{(2)}/\mathbb{Q})$ correspond bijectively to quadratic extensions of $\mathbb{Q}$. But there is only countably many of them (they correspond bijectively to $\mathbb{Q}^\times/\mathbb{Q}^{\times 2} = \langle -1, p_1, p_2, \ldots \rangle \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2$). And there is uncountably many subgroups of index 2 in $\mathrm{Gal}(\mathbb{Q}^{(2)}/\mathbb{Q})$!

1.7 Let $\overline{A} := A \otimes_k \overline{k}$.

(a) If $\overline{A} = \overline{k}[G]$, then A clearly satisfies the given conditions. Suppose now that A satisfies the given conditions. Let $n := \#G$, $\overline{A} = \prod_{i=1}^{n} \overline{k}$ and let $e_1, \ldots, e_n$ be the orthogonal idempotents. Fix a $g \in G$ and note that $ge_1, \ldots, ge_n$ are again orthogonal idempotents. Indeed,

$$(ge_i) \cdot (ge_j) = g(e_i e_j) = \begin{cases} ge_i, & i = j \\ 0, & i \neq j. \end{cases}$$

Thus (from the uniqueness of n-tuples of orthogonal idempotents) g permutes the orthogonal idempotents: $ge_i = e_{\sigma_g(i)}$ for some $\sigma_g \in S_n$. Consider the element $x := \sum_{g \in G} ge_1$. Clearly, $x \in \overline{A}^G = \overline{k}$ and thus

$\sum_{g\in G} e_{\sigma_g(1)} = c = \sum_{i=1}^n ce_i$. However, this means that $c = 1$ and $\{ge_1 : g \in G\} = \{e_1, \ldots, e_n\}$. Consider the homomorphism of G-algebras:

$$\overline{k}[G] \to A, \qquad \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g g \cdot e_1.$$

From the above considerations, it is surjective and is between vector spaces of the same dimension. Thus it is an isomorphism.

(b) *(I believe that the action of G and* Gal(k) *have to* ***commute****).*

We have to show that for an étale k-algebra A, $Hom_k(A, k_s)$ is a transitive G-set if and only if A is a Galois G-algebra. Note that if G acts on $Hom_k(A, k_s)$, then G acts on A. Indeed, then $\overline{A} \cong \overline{A}^{**} = Hom_k(A, \overline{k})^*$ and thus G acts $\overline{k}$-linearly on $\overline{A}$. Since actions of G and Gal(k) commute (where the action on $\overline{A}$ is given by $\sigma(a \otimes x) := a \otimes \sigma(x)$ – so that $\overline{A}^{\mathrm{Gal}(k)} = A$), the action of G descends to A: if $a \in A$ then for any $\sigma \in \mathrm{Gal}(k)$, $g \in G$:

$$\sigma(g(a \otimes 1)) = g(\sigma(a \otimes 1)) = g(a \otimes 1)$$

and thus $g(a \otimes 1) \in \overline{A}^{\mathrm{Gal}(k)} = A$.

From (a) one easily sees that a G-algebra is Galois if and only if G acts simply transitively on the maximal set of orthogonal idempotents. But $Hom_k(A, k_s) \cong Hom_{\overline{k}}(\overline{A}, \overline{k}) = \{e_1^*, \ldots, e_n^*\}$, where (e_i^*) is the dual basis the the orthogonal idempotents. But G acts simply transitively on (e_i^*) if and only if it acts simply transitively on (e_i). This ends the proof.

1.8 (a) Note that if Gal(k) acts on S via even permutations then it acts on $\Delta(S)$ trivially. But the trivial Gal(k)-set on two elements clearly corresponds to $k \times k$.

Suppose now that action of at least one element of Gal(k) induces an odd permutation of elements in S. Then G acts non-trivially on $\Delta(S)$ and therefore (since $\#\Delta(S) = 2$) it is a transitive Gal(k)-set of order 2. Thus by Galois correspondence, it corresponds to some extension of k of degree 2.

(b) Let $\alpha_1, \ldots, \alpha_n$ be the roots of f. Note that then we can identify $S \cong Hom_k(A, k_s)$ with $\{\alpha_1, \ldots, \alpha_n\}$ (as Gal(k)-sets). Note that $d(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2$ (it is the discriminant of f). Thus $\sqrt{d(f)} = \prod_{i<j}(\alpha_i - \alpha_j)$. [1] It is straightforward that for any $g \in \mathrm{Gal}(k)$ we have:

$$g(\sqrt{d(f)}) = \begin{cases} \sqrt{d(f)}, & \text{if } g \text{ acts as an even permutation on } S, \\ -\sqrt{d(f)}, & \text{if } g \text{ acts as an odd permutation on } S. \end{cases}$$

This shows that $\mathrm{Hom}_k(\Delta(A), k_s) \cong \{-\sqrt{d(f)}, \sqrt{d(f)}\}$ (isomorphism of Gal(k)-sets) is isomorphic as a Gal(k)-set to $\Delta(S)$. This ends the proof.

2 Fundamental groups in topology

2.1

Lemma If Y is a Hausdorff space and $x_1, \ldots, x_n \in Y$ are pairwise different, then there exist paiwisely disjoint open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$.
Proof: By the Hausdorff property, we may find open sets V_{ij} such that $x_i \in V_{ij}$, $V_{ij} \cap V_{ji} = \varnothing$. It suffices to take:

$$U_i := \bigcap_{\substack{j=1 \\ j\neq i}}^{n} V_{ij}.$$

Let $G = \{g_1, \ldots, g_n\}$ and take any $y \in Y$. By lemma, we may find pairwise disjoint U_i such that $g_i \cdot y \in U_i$ (note that by assumption, $g \cdot y_1, \ldots, g_n \cdot y$ are pairwise dijoint). Define:

$$U := \bigcap_{i=1}^{n} g_i^{-1} U_i.$$

Then clearly, $y \in U$ and $g_i U \cap g_j U \subset U_i \cap U_j = \varnothing$ for $i \neq j$.

[1] we choose one root, another is $-\prod_{i<j}(\alpha_i - \alpha_j)$

2.2

(Should be (?): ... if and only if the natural map

$$Y \times G \to Y \times_X Y, \qquad (y, g) \mapsto (y, g \cdot y)$$

is a homeomorphism.)

Recall that

$$Y \times_X Y = \{(y_1, y_2) \in Y \times Y : \pi(y_1) = \pi(y_2)\}.$$

Suppose that $\pi : Y \to X$ is a Galois cover. Then G acts freely and transitively on every fiber, in particular for any y_1, y_2 such that $\pi(y_1) = \pi(y_2)$ we may define $y_2 \cdot y_1^{-1} \in G$ (in other words, fibers are principal homogeneous spaces under G). Thus we have mutually inverse homeomorphisms:

$$\begin{array}{rcl} Y \times_X Y & \cong & Y \times G \\ (y_1, y_2) & \mapsto & (y_1, y_2 \cdot y_1^{-1}) \\ (y_1, g \cdot y_1) & \leftarrow\!\shortmid & (y_1, g). \end{array}$$

Suppose now that the natural map

$$Y \times G \to Y \times_X Y, \qquad (y, g) \mapsto (y, g \cdot y)$$

is a homeomorphism. Then G clearly acts transitively on fibers of π and thus $\pi : Y \to X$ is Galois.

2.3 (a) We want to show that $p_* : \pi_1(Y, y) \to \pi_1(X, x)$, $[\gamma] \mapsto [p \circ \gamma]$ is an injection. Suppose that $[\gamma] \in \ker p_*$. Let $f : [0,1] \times [0,1] \to X$, $f_0 = p \circ \gamma$, $f_1 = \iota_x$ (the constant path at x) and suppose that f keeps the endpoints fixed. Use the homotopy lifting property for coverings: to obtain a homotopy $\tilde{f} : [0,1] \times [0,1] \to Y$ lifting f such that $\tilde{f}_0 = \gamma$. Note that $\tilde{f}_1$ is a path lifting ι_x, thus we must have $\tilde{f}_1 = \iota_y$. It is easy to check that $\tilde{f}$ must keep the endpoints fixed. This yields: $[\gamma] = 0$.

(b) By the universal property of $\pi : \widetilde{X}_x \to X$:

$$Fib_x(Y) \cong Hom(\widetilde{X}_x, Y).$$

Thus $y \in p^{-1}(x)$ corresponds to a map $\tilde{p} : \widetilde{X}_x \to Y$ such that $\tilde{p}(\tilde{x}) = y$ and $\pi = p \circ \tilde{p}$. The last equality implies that $\tilde{p}$ is a covering of Y (by Lemma 2.2.11). Note that $\widetilde{X}_x$ is a universal cover of X and thus it is simply connected. But this means that $\tilde{p} : \widetilde{X}_x \to Y$ is the universal covering of Y. Therefore $\pi_1(Y, y)^{op} \cong Aut(\widetilde{X}_x/Y) \hookrightarrow Aut(\widetilde{X}_x/X)$ and (since universal cover is always normal) $Y \cong Aut(\widetilde{X}_x/Y) \backslash \widetilde{X}_x \cong \pi_1(Y, y)^{op} \backslash \widetilde{X}_x$.

2.4 (a) Let $m : G \times G \to G$ be the multiplication map. Note that $\widetilde{G}_e \times \widetilde{G}_e$ is simply connected and thus may be identified with the universal cover $(\widetilde{G \times G})_{(e,e)}$.

We want to find such a group law $\tilde{m} : \widetilde{G}_e \times \widetilde{G}_e \to \widetilde{G}_e$ that the diagram:

$$\begin{array}{ccc} \widetilde{G}_e \times \widetilde{G}_e & \xrightarrow{\tilde{m}} & \widetilde{G}_e \\ \downarrow{\scriptstyle \pi \times \pi} & & \downarrow{\scriptstyle \pi} \\ G \times G & \xrightarrow{m} & G \end{array}$$

is commutative (this is equivalent to π being a homomorphism). In order to do this, it suffices to choose (using the universal property of the universal cover $\widetilde{G}_e$) the unique $\tilde{m} : \widetilde{G}_e \times \widetilde{G}_e \to \widetilde{G}_e$ lifting

$$m \circ (\pi \times \pi) : \widetilde{G}_e \times \widetilde{G}_e \to G$$

such that $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$.

(b) Suppose that $\tilde{m}' : \widetilde{G}_e \times \widetilde{G}_e \to \widetilde{G}_e$ also satisfies the assumptions. Then $\tilde{m}'$ lifts $m \circ (\pi \times \pi)$ (from the above commutative diagram) and $\tilde{m}'(\tilde{e}) = \tilde{e}$. Thus $\tilde{m}' = \tilde{m}$, since there exists a unique such lift.

(c) By the definition of a cover, the map $\pi : \widetilde{G}_e \to G$ is surjective. Note that $\ker \pi = \pi^{-1}(e)$ and we have a bijection of groups

$$\begin{array}{rcl} \pi^{-1}(e) & \cong & Aut(\widetilde{X}_x/X) \\ f(\tilde{e}) & \leftarrow\!\shortmid & f \end{array}$$

(which is an isomorphism of abstract groups, by functoriality of this isomorphism) and moreover $Aut(\widetilde{X}_x/X) \cong \pi_1(X, x)^{op}$. This ends the proof.

2.6 (a) Let $p : \widetilde{X} \to X \times X$, $p_\Delta : \widetilde{X}_\Delta \to X$. By the set-theoretical description of pullback:

$$\begin{aligned}\widetilde{X}_\Delta &= \{([\gamma], x) \in \widetilde{X} \times X : p([\gamma]) = \Delta(x)\}\\ &= \{([\gamma], x) \in \widetilde{X} \times X : (\gamma(0), \gamma(1)) = (x, x)\\ &= \{[\gamma] \in \widetilde{X} : \gamma(0) = \gamma(1)\},\end{aligned}$$

which easily yields the identification $p_\Delta^{-1}(x) = \pi_1(X, x)$.

Monodromy action: let $[\omega] \in \pi_1(X, x)$ and $[\gamma] \in p_\Delta^{-1}(x) = \pi_1(X, x)$. Consider the path $f : [0, 1] \to \widetilde{X}_\Delta$, given by:

$$f(t) = [\omega(tx) \bullet \gamma(x) \bullet \omega^{-1}(tx)].$$

Note that for a fixed t, the endpoints of $f(t) \in \widetilde{X}_\Delta$ are $(\omega(t), \omega(t))$ and thus $p_{\Delta,*}f = \omega$. Moreover, $f(0) = [\gamma]$ and $f(1) = [\omega \bullet \gamma \bullet \omega^{-1}]$. Thus the monodromy action of ω on γ is

$$\omega \bullet \gamma \bullet \omega^{-1}.$$

(cf. also ex. 7(b)).

(b) Let $Y \to [0, 1]$ be an arbitrary cover. We want to explicite the isomorphism $Y_0 \cong Y_1$. Let $w : [0, 1] \to [0, 1]$ be the identity path. Then for any $a \in Y_0$, we may define $w \star a \in Y_1$ by taking $\widetilde{w} : [0, 1] \to Y$ to be the unique lift of w such that $w(0) = a$ and defining $w \star a := \widetilde{w}(1)$ (the "path-monodromy action"). It is immediate that $a \mapsto w \star a$ gives an isomorphism $Y_0 \cong Y_1$ (the inverse map is $b \mapsto w^{-1} \star b$).

In our case:

$$Y = f_*\widetilde{X}_\Delta = \{([\gamma], t) \in \widetilde{X}_\Delta \times [0, 1] : \gamma(0) = \gamma(1) = f(t)\}.$$

Fix $[\gamma] \in (f_*\widetilde{X}_\Delta)_0$. Then $\widetilde{w}$ (as above) is given by the formula $\widetilde{w} : [0, 1] \to Y$:

$$\widetilde{w}(t) := ([f(tx) \bullet \gamma(x) \bullet f(tx)^{-1}], t).$$

Indeed, for a fixed t, $\widetilde{w}(t)$ is a loop in X with the endpoint being $f(t)$. Thus under the isomorphism $Y_0 \cong Y_1$, γ maps to

$$\widetilde{w}(1) = ([f \bullet \gamma \bullet f^{-1}], 1).$$

This ends the proof.

2.7 (a) The map m is the composition of paths, i is the inverse path, $e : X \to \widetilde{X}$ is the constant path at a given point. The listed properties are immediate.

(b) Let $\pi : \widetilde{X} \to X$. Fix any $[\gamma] \in \pi^{-1}((x, x))$, i.e. a loop on X at x. Consider the path $f : [0, 1] \to \widetilde{X}$, where $f(t) \in \widetilde{X}$ is the path

$$f(t)(x) = [\gamma_1(tx) \bullet \gamma(x) \bullet \gamma_2(tx)^{-1}].$$

with beggining and end given by $\gamma_1(t)$ and $\gamma_2(t)$ respectively. Then $\pi \circ f = (\gamma_1, \gamma_2)$ and thus it lifts the path (γ_1, γ_2). Therefore the effect of the monodromy action of (γ_1, γ_2) on $[\gamma] = f(0)$ is

$$f(1) = [\gamma_1(x) \bullet \gamma(x) \bullet \gamma_2(x)^{-1}].$$

This ends the proof.

(c) Let us introduce some notation. Suppose that $g : Y \to Z$ is a cover. Denote:

- for any path $\gamma : [0, 1] \to Z$, $z := \gamma(0) \in Z$, $\widetilde{z} \in g^{-1}(z)$, let $\gamma \star \widetilde{z} := \widetilde{\gamma}(1)$, where $\widetilde{\gamma} : [0, 1] \to Y$ is the unique lift of gamma such that $\widetilde{\gamma}(0) = \widetilde{z}$, ("path-monodromy action")
- i_a – constant path at a,
- m_Π, etc – grupoid operation on Π.

Let $\Pi \to X \times X$ be a grupoid cover. Define $\Phi : \widetilde{X} \to \Pi$ by:

$$\Phi([\gamma]) := (i_{\gamma(0)}, \gamma) \star e_\Pi(\gamma(0))$$

(note that paths on $X \times X$ are pairs of paths on X). We have to check that it is compatible with $m_\Pi : \Pi \times_X \Pi \to \Pi$.

Note that the "path-monodromy action" of $X \times X$ on $\Pi \times_X \Pi$ is given by:

$$(\gamma_1, \gamma_2) \star (x_1, x_2) = \Big((\gamma_1, i_{b_1}) \star x_1, (i_{a_2}, \gamma_2) \star x_2 \Big)$$

(where $(s(x_i), t(x_i)) = (a_i, b_i)$). Note that $m_\Pi : \Pi \times_X \Pi \to \Pi$ is a map of $X \times X$ spaces and commutes with the "path-monodromy action" of $X \times X$, i.e.:

$$(\gamma_1, \gamma_2) \star m(x_1, x_2) = m\Big((\gamma_1, \gamma_2) \star (x_1, x_2) \Big) = m\Big((\gamma_1, i_{b_1}) \star x_1, (i_{a_2}, \gamma_2) \star x_2 \Big). \quad (*)$$

Therefore:

$$\begin{aligned}
m_\Pi(\Phi([\gamma_1]), \Phi([\gamma_2])) &= m_\Pi\Big((i_{\gamma_1(0)}, \gamma_1) \star e_\Pi(\gamma_1(0)), (i_{\gamma_2(0)}, \gamma_2) \star e_\Pi(\gamma_2(0)) \Big) (\text{by } (*)) \\
&= (\gamma_1, \gamma_2) \star m_\Pi\Big((\gamma_1^{-1}, \gamma_1) \star e_\Pi(\gamma_1(0)), e_\Pi(\gamma_2(0)) \Big) \; (\text{since } m \circ (e, id) = id) \\
&= (\gamma_1, \gamma_2) \star \Big((\gamma_1^{-1}, \gamma_1) \star e_\Pi(\gamma_1(0)) \Big) \\
&= \Big((\gamma_1, \gamma_2) \bullet (\gamma_1^{-1}, \gamma_1) \Big) \star e_\Pi(\gamma_1(0)) \\
&= (i_{\gamma_1(1)}, \gamma_2 \bullet \gamma_1) \star e_\Pi(\gamma_1(0)) \\
&= \Phi([\gamma_2 \bullet \gamma_1])
\end{aligned}$$

Remark: *this is small mistake – the result should be the equality* $m_\Pi(\Phi([\gamma_1]), \Phi([\gamma_2])) = \Phi([\gamma_1 \bullet \gamma_2])$. *In order to correct it one should trace the whole proof.*

2.9 We'll show that the necessary and sufficient condition is the irreducibility of X. Let $s_1, s_2 \in S$, $s_1 \neq s_2$.

($\Leftarrow$) Suppose that $\mathcal{F}$ is a sheaf and that X is reducible, i.e. that there exist open $U_1, U_2 \subset X$, which are disjoint. Consider $(s_1, s_2) \in \mathcal{F}(U_1) \times \mathcal{F}(U_2)$. Then $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$, since $U_1 \cap U_2 = \varnothing$. Thus there should exist $s \in \mathcal{F}(U_1 \cup U_2)$ such that $s|_{U_i} = s_i$. This is however impossible, since $u_1 \neq u_2$ and the restricition maps are identity maps.

($\Rightarrow$) Suppose that X is irreducibile, i.e. that any two open subsets of X have non-empty intersection. Suppose that $(s_i)_i \in \mathcal{F}(U_i)$ and that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then (since $U_i \cap U_j \neq \varnothing$ and the restricition maps are identities) $s_i = s_j$. Thus (s_i) glues to a (unique) element $s = s_i \in S = \mathcal{F}(\bigcup_i U_i)$. This proves the sheaf property.

2.10 We want to show that

$$\begin{array}{rcl}
\{p : Y \to X - \text{local homeomorphisms}\} & \leftrightarrow & \{\text{ sheaves on } X\} \\
(p : Y \to X) & \mapsto & Sec(p) \\
(X_\mathcal{F} \to X) & \leftarrow\!\shortmid & \mathcal{F}
\end{array}$$

(here $Sec(p)$ is the sheaf of local sections of p) is an equivalence of categories.

- **Step I:** for any sheaf $\mathcal{F}$, $X_\mathcal{F} \to X$ is a local homeomorphism.

  Proof: Let $s \in X_\mathcal{F}$, $s = [(U, t)] \in \mathcal{F}_x = \lim_{\leftarrow} \mathcal{F}(V)$. Let also $i_t : U \to X_\mathcal{F}$ be the map associated to t, as defined in text. Then $i_t(U)$ is an open set and moreover $p|_{i_t(U)}$ is a local homeomorphism – indeed, $p|_{i_t(U)} \circ i_t = id$, $i_t \circ p|_{i_t(U)} = id$.

- **Step II:** For any sheaf $\mathcal{F}$, $Sec(X_\mathcal{F} \to X) = \mathcal{F}$.

  Proof: let $p : U \to X_\mathcal{F}$ be a continuous section of $X_\mathcal{F} \to X$. Fix a point $Q \in X$ and let $p(Q) = [(f_Q, U_Q)] \in \mathcal{F}_Q \subset X_\mathcal{F}$.

  Claim: There exists W_Q such that $p|_{W_Q} = i_{f_Q}|_{W_Q}$.

  Proof of the claim: By the definition of topology on $X_\mathcal{F}$, $i_{f_Q}(U_Q)$ is open. Since p is continuous at Q, there exists an open set $W_Q \subset X$ such that $p(W_Q) \subset i_{f_Q}(U_Q)$. But p is a section of $X_\mathcal{F} \to X$ and $i_{f_Q}(U_Q)$ contains only one point in fiber over each $R \in U_Q$, namely $i_{f_Q}(R)$.

("if image of one section is contained in the image of another section, they must be equal")

The Claim easily implies Step II. Indeed, if we show that for $Q_1 \neq Q_2$,

$$f_{Q_1}|_{W_{Q_1}\cap W_{Q_2}} = f_{Q_2}|_{W_{Q_1}\cap W_{Q_2}} \qquad (*)$$

then by the sheaf property we may glue $f_Q \in \mathcal{F}(W_Q)$ to $f \in \mathcal{F}(U)$ and then $p = i_f$. But

$$i_{f_{Q_1}}|_{W_{Q_1}\cap W_{Q_2}} = p|_{W_{Q_1}\cap W_{Q_2}} = p|_{W_{Q_1}\cap W_{Q_2}} = i_{f_{Q_1}}|_{W_{Q_1}\cap W_{Q_2}}$$

easily implies $(*)$.

- **Step III:** For any local homeomorphism $p : Y \to X$, $X_{Sec(p)} \cong Y$.

  Proof: note that for every $Q \in Y$ there exists a neighbourhood $U_Q \subset Y$ of Q such that $p|_{U_Q}$ is a homeomorphism. Let $f_Q := (p|_{U_Q})^{-1}$. Then we obtain homeomorphisms:

$$\begin{array}{rcl} Y & \cong & X_{Sec(p)} \\ Q & \mapsto & [(f_Q, U_Q)] \in Sec(p)_{p(Q)} \\ f(P) & \leftarrow\!\shortmid & [(f, U)] \in Sec(p)_P \end{array}$$

2.11 Note that any map $f : \mathcal{F} \to \mathcal{G}$ induces a map on stalks, which in turn induces a map $f' : X_{\mathcal{F}} \to X_{\mathcal{G}}$. Thus we obtain a (unique) map $\tilde{f} : Sec(X_{\mathcal{F}}) \to Sec(X_{\mathcal{G}})$, $\tilde{f}(s) = f' \circ s$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{G} \\ \downarrow & & \downarrow \\ Sec(X_{\mathcal{F}}) & \xrightarrow{\tilde{f}} & Sec(X_{\mathcal{G}}). \end{array}$$

But in the previous problem we've proven that (since $\mathcal{G}$ is **a sheaf** and not merely a presheaf) the natural map $\mathcal{G} \to Sec(X_{\mathcal{G}})$ is an isomorphism. This ends the proof.

3 Riemann surfaces

3.1 *(the map should be proper in order for the problem to make sense)*

By Proposition 3.2.1, for every $y \in \varphi^{-1}(x)$ there exists a neighbourhood U_y of y and local coordinates at y, x (which we will both denote as z by abusing the notation) such that $\varphi|_{U_y}$ is in these local coordinates

$$z \mapsto z^{e_y}.$$

Let $U := \bigcap_{y\in\varphi^{-1}(x)} U_x$ (note that this is an open set, since this is a finite intersection). Pick any $w \in U$. On one hand, $\#\varphi^{-1}(w) = n$, since U does not contain branch points and φ is a cover of degree n outside of the set of branch points. On the other hand, since in the neighbourhood of $y \in \varphi^{-1}(x)$, φ acts as $z \mapsto z^{e_y}$, w has e_y preimages in a neighbourhood of y. In total it has

$$\#\varphi^{-1}(w) = \sum_{y\in\varphi^{-1}(x)} e_y$$

preimages. By comparing both results we prove the theorem.

3.2 (a) By Corollary 3.3.12:

$$Hom(\mathbb{P}^1(\mathbb{C}), \mathbb{P}^1(\mathbb{C})) \cong Hom(\mathcal{M}(\mathbb{P}^1(\mathbb{C})), \mathcal{M}(\mathbb{P}^1(\mathbb{C}))) = Hom(\mathbb{C}(t), \mathbb{C}(t)).$$

But by Lüroth theorem, every subfield of $\mathbb{C}(t)$ is of the form $\mathbb{C}(f(t))$ for some $f \in \mathbb{C}(t)$. This ends the proof.

(b) We'll prove that

$$e_y = \begin{cases} ord_y\, f', & y \text{ is not a pole of } f, \\ ord_y\, (1/f)', & y \text{ is a pole of } f. \end{cases}.$$

In particular, the branch points are the zeroes of f' and zeroes of $(1/f)'$.

Without loss of generality (by applying the automorphism of $\mathbb{P}^1(\mathbb{C})$, $z \mapsto 1/z$) we may assume that y is not a pole of f. Then the local coordinates at y and $f(y)$ are obviously $z-y$ and $z-f(y)$ respectively. It suffices to note that $(z-f(y)) \circ f = f - f(y) = (z-y)^e \cdot h(z)$, where $e = ord_y f'$ and $h(y) \in \mathbb{C}^\times$.

(c) Since f is an automorphism, it is in particular a bijection and thus it has a unique pole and a unique zero. Moreover, by Riemann-Hurwitz formula, $\sum_y (e_y - 1) = 0$ and thus by previous part, both the zero and the pole are simple. But all rational functions with a single pole and a single zero are of the form $\frac{at+b}{ct+d}$.

(d) Without loss of generality, suppose $f(t) = \frac{at+b}{ct+d}$, $c \neq 0$. The equation $f(t) = t$ is a quadratic equation and thus has 1 or 2 solutions.

3.3 (a) **Remark 1:** I think one has to assume that X is connected.

Remark 2: note that Theorem 3.2.7 easily implies that $Y' \to X'$ is a trivial cover iff $Y \to X$ is a trivial cover, i.e. $Y = Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_r$ and $p|_{Y_i} : Y_i \to X$ is a homoeomorphism.

Solution: Let $Y = Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_r$ be the decomposition into connected components. All of them are closed subsets of Y and thus compact. One easily checks that $\pi_i : Y_i \to X$ are branched coverings and $\mathcal{M}(Y) \cong \prod_i \mathcal{M}(Y_i)$. This easily implies that $\mathcal{M}(Y)$ is a product of $\mathcal{M}(X)$ iff $\mathcal{M}(Y_i) \cong \mathcal{M}(X)$ as $\mathcal{M}(X)$-algebras for all i. But by equivalence of categories $\mathcal{M}(Y_i) \cong \mathcal{M}(X)$ holds iff π_i is an isomorphism.

(b) First we prove an analogue of Exercise 2.2 for fields:

Lemma Let K be a perfect field and let L be its finite extension. Then L/K is Galois iff the étale K-algebra $L \otimes_K L$ is isomorphic to a finite product of copies of K.
Proof: let $L = K[x]/(f(x))$ and let $f = \prod_j f_j$ be decomposition into irreducible factors over L. Then by Chinese Remainder Theorem:

$$L \otimes_K L = K[x]/(f(x)) \otimes_K L = L[x]/(f(x)) = \prod_j L[x]/(f_j(x)).$$

Note that for all j, $L[x]/(f_j(x))$ is a field, since f_j is irreducible over L and thus (f_j) is a prime (and maximal, since $L[x]$ is a PID) ideal.
But $L[x]/(f_j) \cong K$ iff $\deg f_j = 1$. Thus $L \otimes_K L$ is isomorphic to a finite product of copies of K iff $\deg f_j = 1$, which is equivalent to L being a Galois extension of K.

Lemma $Y \times_X Y \to Y$ is a branched cover and $Y' \times_{X'} Y \to Y'$ is a cover. Moreover:

$$\mathcal{M}(Y \times_X Y) \cong \mathcal{M}(Y) \otimes_{\mathcal{M}(X)} \mathcal{M}(X)$$

Proof: "left as an exercise".

Now, $Y' \to X'$ is a Galois cover iff $Y' \times_{X'} Y$ is a trivial cover iff (by (a)) $\mathcal{M}(Y \times_X Y) \cong \mathcal{M}(Y) \otimes_{\mathcal{M}(X)} \mathcal{M}(X)$ is isomorphic to a finite product of copies of $\mathcal{M}(X)$ iff (by Lemma) $\mathcal{M}(Y)/\mathcal{M}(X)$ is a Galois extension.

3.4 Suppose to the contrary that there is only one branch point. Then from the Riemann-Hurwitz formula:

$$2g_Y - 2 = d \cdot (2 \cdot 0 - 2) + \sum_{Q \in \pi^{-1}(P)} (e_P - 1).$$

On the other hand, $\sum_{Q \in \pi^{-1}(P)} e_P = d$. Thus:

$$\sum_{Q \in \pi^{-1}(P)} (e_P - 1) = d - \#\pi^{-1}(P)$$

and

$$g_Y = 1 - d + \frac{d - \#\pi^{-1}(P)}{2} = 1 - \frac{d + \#\pi^{-1}(P)}{2}.$$

Since $g_Y \geqslant 0$, we must have $d = \#\pi^{-1}(P) = 1$. But this means that P is **not** a branch point, and $Y \to X$ is a cover of degree 1, i.e. an isomorphism! This ends the proof.

3.5 **From errata:** the problem should be formulated like this:

Let $n \in \mathbb{Z}_+$ and consider the dihedral group D_n. Show that every complex torus $X = \mathbb{C}/\Lambda$ has a Galois branched cover $Y \to X$ with group D_n, which:

- has 4 branch points when $n = 2m$ is even (each of ramification index m)
- has 2 branch points when n is odd (each of ramification index n).

Remark: In fact, the two cases follow from the fact that:

$$D_n' = \langle \sigma^2 \rangle \cong \begin{cases} \mathbb{Z}/m, & n = 2m \\ \mathbb{Z}/n, & 2 \nmid n. \end{cases}$$

Solution: Let $P \in X$ be a point in the middle of the fundamental parallelogram. Let also $X' := X \backslash \{P\}$. First, we'll show that $\pi_1(X', x) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$.

Indeed, note that X' is homotopic to $S^1 \wedge S^1$. To show it, consider the fundamental parallelogram of X with one point P in the interior removed. Then we can retract the rest of the parallelogram by projecting from P onto the perimeter of the parallelogram. But the parallelogram is clearly homeomorphic to $S^1 \wedge S^1$. By Seifert-van Kampfen theorem, $\pi_1(S^1 \wedge S^1, x) \cong \mathbb{Z} * \mathbb{Z}$.

To recover the ramification indices of the cover we will need the following lemma:

Lemma Let X be a connected compact Riemann surface, $S \subset X$ – a finite set and $X' := X \backslash S$. Let $Y' \to X'$ be a topological cover and $\pi : Y \to X$ – the corresponding (by Theorem 3.2.7) branched cover of Riemann surfaces. Let Z be the $\pi_1(X', x)$-set corresponding to the cover $Y' \to X'$.
Fix any $P \in S$ and let $[\gamma] \in \pi_1(X', x)$ be a loop around P. The following quantities are equal:

- set of ramification indices $\{e_Q : Q \in \pi^{-1}(P)\}$ (correspondingly $\#\pi^{-1}(P)$),
- $\{n_1, \ldots, n_r\}$ (correspondingly r), where the cover $Y' \to X'$ restricted to γ is of the form $Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_r \to X'$, where Y_i is a connected n_i-fold cover of the circle γ,
- lengths of orbits of Z under the action of $\langle \gamma \rangle$ (correspondingly number of orbits).

Proof: follows at once from the construction of Y from Y'.

The epimorphism

$$\begin{array}{rcl} \pi_1(X', x) \cong \mathbb{Z} * \mathbb{Z} & \longrightarrow & D_n \\ a & \mapsto & \sigma \\ b & \mapsto & \tau \end{array}$$

corresponds to a D_n Galois cover $Y' \to X'$. By Theorem 3.2.7 this extends uniquely to a branched cover $Y \to X$ of connected compact Riemann surfaces. Let Z be the $\pi_1(X', x)$-set corresponding to the cover $Y' \to X'$; we can identify Z with D_n.

The loop around P will be $aba^{-1}b^{-1} \in \pi_1(X', x)$. Note that $aba^{-1}b^1$ acts on Z as

$$\sigma\tau\sigma^{-1}\tau^{-1} = \sigma^2.$$

We will consider now two cases, corresponding to parity:

- $n = 2m$:

  Then there are four orbits of action of σ^2 on $Z = D_n$:

  $$\{e, \sigma^2, \sigma^4, \sigma^6, \ldots, \sigma^{2m-2}\}, \quad \{\sigma, \sigma^3, \sigma^5, \sigma^7, \ldots, \sigma^{2m-1}\},$$
  $$\{\tau, \tau\sigma^2, \tau\sigma^4, \ldots, \tau\sigma^{2m-2}\}, \quad \{\tau\sigma, \tau\sigma^3, \tau\sigma^5, \tau, \ldots, \tau\sigma^{2m-1}\}.$$

  each of cardinality m.

- $2 \nmid n$:
  Then there are two orbits of action of σ^2 on $Z = D_n$ (since $\langle \sigma^2 \rangle = \langle \sigma \rangle$):
  $$\{e, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}, \quad \{\tau, \tau\sigma, \tau\sigma^2, \ldots, \tau\sigma^{n-1}\}.$$
  each of cardinality n.

Using the above lemma, this ends the proof.

3.6 (a) The Riemann-Hurwitz formula with $g_X = g_Y = 1$ becomes:
$$0 = 0 + \sum_{P \in X} (e_P - 1)$$
and thus $e_P = 1$ for every $P \in X$.

(b) Let $\widetilde{b} \in \mathbb{C}$ be an arbitrary lift of $b := \phi(0 + \Lambda) \in \mathbb{C}/\Lambda'$. Consider the diagram:
$$\begin{array}{ccc} & & \widetilde{X} = \mathbb{C} \\ & \overset{\widetilde{\phi}}{\nearrow} & \downarrow \\ Z = \mathbb{C} & \xrightarrow{\phi \circ p} & X = \mathbb{C}/\Lambda' \end{array}$$
Since $\widetilde{X} \to X$ is the universal cover, there exists a unique lift of $\phi \circ p : Z = \mathbb{C} \to X = \mathbb{C}/\Lambda'$ to $\widetilde{\phi} : Z = \mathbb{C} \to \widetilde{X} = \mathbb{C}$.

(c) & (d) It is easier to prove directly that $\widetilde{\phi}(z) = az + b$ than to follow the path suggested by Szamuely. Note that for any $l \in \Lambda$, $\widetilde{\phi}(z + l) - \widetilde{\phi}(z) \in \Gamma'$. Therefore, since Γ' is discrete and $\widetilde{\phi}$ continuous:
$$\widetilde{\phi}(z + l) - \widetilde{\phi}(z) \equiv c(l) \quad (*)$$
for some $c(l) \in \Gamma'$. Let $\Gamma = \mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2$. By using $(*)$ for τ_i and differentiating it, we see that $\widetilde{\phi}' : \mathbb{C} \to \mathbb{C}$ is a holomorphic, doubly periodic function. Therefore it is bounded (since its values are determined by the values on the fundamental domain of $\mathbb{C}/\Lambda$) and must be constant by Liouville theorem. This immediately implies that $\widetilde{\phi}(z) = az + b$. One easily sees that $\widetilde{\phi}$ is induced by map of tori iff $a\Lambda + b \subset \Lambda'$.

3.A Let X be a connected and locally simply connected topological space and let $p : Y \to X$ be a connected cover of degree d. Show that:

(a) p is Galois, if $d = 2$.

(b) There is an explicit example of a connected cover $p : Y \to X$ such that $d = 3$ and p is not Galois.

(c) If $d = 3$ and p is not Galois then $Aut(Y/X) = \{id_Y\}$.

Solution:

(a) We have to show that $Aut(Y/X)$ acts transitively on fibers. We construct an element $\varphi \in Aut(Y/X)$ in the following way: for any $y \in Y$, let $x := p(y) \in X$ and let $p^{-1}(x) = \{y, y'\}$. Then we put $\varphi(y) := y'$. It suffices to check that φ defined in this way is continuous. With the above notation, we may choose a neighbourhoods U, U', V of y, y', x respectively such that $p|_U : U \to V$ and $p|_{U'} : U' \to V$ are homeomorphisms. It is obvious that $\varphi|_V = (p|_{U'})^{-1}$. Thus φ is continuous. From the construction it follows that $\varphi \in Aut(Y/X)$ and that $Aut(Y/X)$ acts transitively on fibers (since φ takes one point in the fiber to the second point).

(b) Consider the map $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, $z \mapsto z^3 + z$. On the level of the function field this corresponds to the extension $\mathbb{C}(t)[z]/(z^3 + z - t)$ of $\mathcal{M}(\mathbb{P}^1(\mathbb{C})) = \mathbb{C}(t)$. But this extension is non-Galois, since the discriminant of the polynomial $z^3 + z - t \in \mathbb{C}(t)[z]$ is not a square (standard Galois theory). It suffices to note that the covering is Galois, if and only if the corresponding extension of function fields is Galois.

(c) Again, it suffices to prove that if L/K is a non-Galois extension of fields of degree 3 then $Aut(L/K) = \{id_L\}$. Let $M := L^{Aut(L/K)}$. Then L/M is a Galois extension of degree $\#Aut(L/K)$. On the other hand: $[L : M] | [L : K] = 3$. Thus, since $M \neq K$, $[L : M] = 1$. This ends the proof.

3.B Let X be a connected and locally simply connected topological space. Let also $x \in X$ and suppose that $\pi_1(X, x)$ is finite. Prove that every continuous map $f : X \to S^1$ is homotopic to a constant map.

Solution: Suppose we are given $f : X \to S^1$. We'll use the following (classical) lifting criterion:

Lemma (the lifting criterion) Let $p : Z \to Y$ be a cover and let $f : X \to Y$ be a continuous map. Let also $x \in X$, $y := f(x) \in Y$ and $z \in p^{-1}(y)$. The lift of f to $\widetilde{f} : X \to Z$ such that $\widetilde{f}(x) = z$ exists iff $f_*(\pi_1(X, x)) \subset p_*(\pi_1(Z, z))$.

In our case take X as in the assumption, $Y = S^1$, $Z = \mathbb{R}$ and $Z \to Y$ – the covering map. Note that $f_*(\pi_1(X, x))$ is a finite group, which is a subgroup of $\mathbb{Z}$ and thus it is trivial. Thus the assuptions of the lifting criterion are satisfied and we have a lift $\widetilde{f} : X \to \mathbb{R}$. $\widetilde{f}$ is clearly null-homotopic, since $\mathbb{R}$ is contractible. Therefore, f is also null-homotopic.

4

5.A **Problem:** Let $X = \operatorname{Spec} \mathbb{Z}_{(p)} = \{\eta, (p)\}$ and consider the maps:

$$i : \operatorname{Spec} \mathbb{F}_p \to X.$$

(a) Prove that $\pi_1(i)$ is injective.

(b) If K is the composite of finite extensions $L/\mathbb{Q}$ such that p splits completely in L then $\operatorname{Gal}(K/\mathbb{Q})$ is the quotient of $\pi_1(X, \overline{x})$ by the normal subgroup generated by $\pi_1(i)(\pi_1(\operatorname{Spec} \mathbb{F}_p))$.

Solution:

(a) Firstly, let us describe the morphism $G_{\mathbb{F}_p} \to \pi_1(X, \overline{x})$. Let $\overline{x}$ be the geometric point $\overline{x} : \operatorname{Spec} \overline{\mathbb{Q}} \to X$, let $\overline{y}$ – the geometric point $\overline{y} : \operatorname{Spec} \overline{\mathbb{F}}_p \to \operatorname{Spec} \mathbb{F}_p$ and $\overline{z}$ – the geometric point $\overline{z} : \operatorname{Spec} \overline{\mathbb{F}}_p \to X$. Note that we have canonical isomorphisms:

$$\pi_1(X, \overline{x}) \cong \operatorname{Gal}(F/\mathbb{Q}), \qquad \pi_1(\operatorname{Spec} \mathbb{F}_p, \overline{y}) \cong \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

(where F is the maximal extension of $\mathbb{Q}$ unramified outside of p) and an isomorphism

$$\pi_1(X, \overline{x}) \cong \pi_1(X, \overline{z}),$$

which depends on the choice of the "path" between $\overline{x}$ and $\overline{z}$. We will now describe this choice. Suppose that $F = \bigcup_n F_n$, where $F_n/\mathbb{Q}$ are finite Galois extensions, $F_0 = \mathbb{Q}$ and $F_n \subset F_{n+1}$. The choice of the path consists of choosing primes $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{F_n}$ such that $\mathfrak{p}_0 = p$, $\mathfrak{p}_{n+1} | \mathfrak{p}_n$. Since $\mathfrak{p}_n$ are unramified, the decomposition groups $D(\mathfrak{p}_n) \subset \operatorname{Gal}(F_n/\mathbb{Q})$ are canonically isomorphic to $\operatorname{Gal}(\mathbb{F}_{\mathfrak{p}_n}/\mathbb{F}_p)$, i.e. we obtain a monomorphism:

$$\operatorname{Gal}(\mathbb{F}_{\mathfrak{p}_n}/\mathbb{F}_p) \to \operatorname{Gal}(F_n/\mathbb{Q}). \qquad (*)$$

Note that $\bigcup_n \mathbb{F}_{\mathfrak{p}_n} = \overline{\mathbb{F}}_p$, as there are finite Galois extensions $L/\mathbb{Q}$ unramified at $\mathfrak{p}|p$ and such that $[\kappa(\mathfrak{p}) : \mathbb{F}_p]$ is arbitrarly large (take for example suitable cyclotomic extensions). Thus by taking the direct limit in $(*)$ we obtain the desired map and we see that it is a monomorphism.

(b) Let N be the largest normal subgroup of $\pi_1(X, \overline{x})$ containing the image of $\pi_1(\operatorname{Spec} \mathbb{F}_p, \overline{y})$. Let

$$N_n := im\Big(N \subset \operatorname{Gal}(F/\mathbb{Q}) \twoheadrightarrow \operatorname{Gal}(F_n/\mathbb{Q})\Big).$$

Note that $F_n^{N_n}$ is (by the very definition) the largest subextension of $F_n/\mathbb{Q}$, which is Galois over $\mathbb{Q}$ and in which the Frobenius acts trivially. But the Frobenius acts trivially in a Galois extension if and only if p splits completely! Thus

$$F^N = \bigcup_n F_n^{N_n}$$

is the largest subextension of $F/\mathbb{Q}$, which is Galois over $\mathbb{Q}$ and in which p splits completely. Equivalently, $F^N = K$ (since every extension in which p splits completely is contained in a Galois extension, in which p splits completely) and $N = \operatorname{Gal}(F/K)$. By the standard properties of Galois groups we obtain the desired short exact sequence:

$$0 \to N = \operatorname{Gal}(K/F) \to \operatorname{Gal}(K/\mathbb{Q}) \to \operatorname{Gal}(F/\mathbb{Q}) \to 0.$$

5.B **Problem:** Let

$$X = Spec(\mathbb{C}[x, y]/(y^2 - x^3))$$

be the cuspidal cubic curve over $\mathbb{C}$. Prove that any finite connected étale cover of X is trivial, i.e. that $\pi_1(X, x) = 0$.

Hint: Consider the normalization of X: $\widetilde{X} \to X$, given by:

$$\phi : \widetilde{X} = Spec(\mathbb{C}[t]) \to X, \quad t \mapsto (t^2, t^3)$$

Show that ϕ is a bijection. Show that for any finite connected étale cover $Y \to X$ the induced base change $Y \times_X \widetilde{X} \to Y$ is a bijection. Deduce that such a Y must be a trivial cover.

Solution: Let

$$\phi : \widetilde{X} = Spec(\mathbb{C}[t]) \to X, \quad t \mapsto (t^2, t^3)$$

be the normalization of X. Note that ϕ is a bijection, since the inverse function might be given by:

$$g(x, y) = \begin{cases} y/x, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(note that g is a morphism on $X \backslash \{(0, 0)\}$, but not on $(0, 0)$). Let $\psi : Y \to X$ be a connected finite étale cover. Consider $\widetilde{Y} := Y \times_X \widetilde{X}$. Note that the natural map $h : \widetilde{Y} \to Y$ is a bijection (on closed points). Indeed:

$$\widetilde{Y}(\mathbb{C}) = \{(\widetilde{x}, y) \in \widetilde{X}(\mathbb{C}) \times Y(\mathbb{C}) : \phi(\widetilde{x}) = \psi(y)\}, \qquad h(\widetilde{x}, y) = y$$

and thus the inverse of h is given by $y \mapsto (g(\psi(y)), y)$. We'll use the following simple lemma:

> **Lemma** Suppose that $h : \widetilde{Y} \to Y$ is a continuous bijection of topological spaces, which is an open map. If Y is connected, then $\widetilde{Y}$ also.
> **Proof:** let $\widetilde{Y} = U_1 \sqcup U_2$ be a decomposition into disjoint open sets. Then $Y = h(U_1) \sqcup h(U_2)$ (note that $h(\widetilde{Y}) = Y$; the sum is disjoint, since h is injective), and since Y is connected, we have $h(U_1) = \varnothing$ or $h(U_2) = \varnothing$. But, since h is a bijection, $U_1 = \varnothing$ or $U_2 = \varnothing$, which ends the proof.

(note that $h : \widetilde{Y} \to Y$ is open, since open sets are precisely the complements of finite sets). Also, since base chenge of an étale morphism is étale, $\widetilde{Y} \to \widetilde{X}$ is a connected finite étale cover of $\widetilde{X} = \mathbb{A}^1_{\mathbb{C}}$. But $\pi_1(\mathbb{A}^1_{\mathbb{C}}) = 0$ and thus $\widetilde{Y} \to \widetilde{X}$ must be a trivial cover, i.e. an isomorphism! This easily implies that $Y \to X$ is a trivial cover. Indeed, it suffices to check that $Y \to X$ is a bijection (on closed points), since a finite étale cover of degree 1 is an isomorphism. Let $x \in X(\mathbb{C})$ and denote $\widetilde{x} := g(x) \in \widetilde{X}(\mathbb{C})$. Then, since $\widetilde{Y} \to \widetilde{X}$ is a bijection, there exists precisely one $y \in Y(\mathbb{C})$ such that $\phi(\widetilde{x}) = \psi(y)$, i.e. $x = \psi(y)$. This ends the proof.

5.C Let $X = \mathrm{Spec}(\mathbb{C}[x, y]/(y^2 - x^3 - x^2))$. Show that

$$\pi_1(X, \eta) \cong \widehat{\mathbb{Z}}.$$

Solution: (loosely based on Jacob Tsimerman's notes: http://www.math.toronto.edu/ jacobt/Lecture7.pdf)

Firstly, we have to find the normalization of X. Blow up X at 0, by setting $y = x \cdot u$ and compute the strict transform:

$$(xu)^2 = x^2 \cdot (x + 1) \quad \Rightarrow \quad u^2 = x + 1.$$

The curve $u^2 = x + 1$ is clearly isomorphic to $\mathbb{A}^1$ via $t \mapsto (u(t), x(t)) := (t, t^2 - 1)$. Finally, we obtain a normalization:

$$\phi : \widetilde{X} := \mathrm{Spec}\, \mathbb{C}[t] \to X, \quad t \mapsto (y(t), x(t)) := (t \cdot (t^2 - 1), t^2 - 1).$$

Let $\psi : Y \to X$ be a connected finite étale cover. Consider $\widetilde{Y} := Y \times_X \widetilde{X}$. Again, as in Problem B, $\pi_1(\widetilde{Y}) = 0$ and thus the cover $\widetilde{Y} \to \widetilde{X}$ must be of the form

$$\widetilde{Y} \cong \bigsqcup_{i=1}^{n} \widetilde{X}_i \to \widetilde{X},$$

where $\widetilde{X}_i$ are copies of $\widetilde{X}$. Denote the embedding $\widetilde{X} \to \bigsqcup_{i=1}^{n} \widetilde{X}_i$ on the i-th component by p_i.

Let $P := (0,0) \in X$, $y_1, \ldots, y_n \in \psi^{-1}(P)$. Note that $\psi^{-1}(P) = \{-1, 1\}$. Note that $\widetilde{X} \to X$ is an isomorphism out of $(0,0) \in X(\mathbb{C})$ and therefore $\widetilde{Y} \to Y$ is an isomorphism out of $y_1, \ldots, y_n$. Moreover, over each y_i there must be two points in $\widetilde{Y}$, that must be of the form $p_{a_i}(-1) \in \widetilde{X}_{a_i}$, $p_{b_i}(1) \in \widetilde{X}_{b_i}$. In other words, the topological space of Y is formed from $\widetilde{Y} \cong \bigsqcup_{i=1}^n \widetilde{X}_i$ by taking a point -1 on each copy of $\widetilde{X}$ and glueing with a point 1 on a different copy of $\widetilde{X}$. Note that Y is connected – thus without loss of generality we may renumerate the copies of $\widetilde{X}$ in such a way that -1 on $\widetilde{X}_i$ is glued to 1 on $\widetilde{X}_{i+1}$ (where we put $\widetilde{X}_{n+1} = \widetilde{X}_1$).

Consider now the scheme formed by glueing $\widetilde{X}$ in such a way. It is given by:

$$Y' := \operatorname{Spec} R_n, \quad R_n := \{(P_1, \ldots, P_n) \in \prod_{i=1}^n \mathbb{C}[t_i] : P_i(-1) = P_{i+1}(1)\}.$$

We'll show that $\Psi : Y' \to X$ is finite étale. Recall that an equivalent definition for this is that $\Psi_* \mathcal{O}_{Y'}$ is a locally free $\mathcal{O}_X$-module (of finite rank) and fiber over each $P \in X$ is a spectrum of a finite étale $\kappa(P)$-algebra.

Lemma The normalization homomorphism:

$$\mathbb{C}[\overline{x}, \overline{y}] := \mathbb{C}[x,y]/(y^2 - x^3 - x^2) \to \mathbb{C}[t], \quad (x,y) \mapsto (t^2 - 1, t \cdot (t^2 - 1))$$

identifies $\mathbb{C}[\overline{x}, \overline{y}]$ with the ring:

$$A := \{f(t) \in \mathbb{C}[t] : f(-1) = f(1)\}.$$

Slogan: „nodal curve is $\mathbb{A}^1$ with -1 and 1 identified".
Proof: ??

Note that out of $(0,0)$, $Y_1 \to X$ is a trivial covering by n copies of $X \backslash \{(0,0)\}$. Thus it suffices to check the neighbourhood of $(0,0)$ (corresponding to $t = \pm 1$). Indeed, let $W_i \in \mathbb{C}[t]$ be arbitrary polynomials such that $W_i(1) = 1$, $W_i(-1) = \zeta_n^i$ ($i = 0, \ldots, n-1$). Let S be the multiplicative set generated by elements W_i. Then $S^{-1} R_n$ is a free $S^{-1} A$-module of rank n with basis given by:

$$b_i := (W_i(t), \zeta_n^i W_i(t), \zeta_n^{2i} W_i(t), \ldots, \zeta_n^{(n-1)i} W_i(t))$$

(one easily checks that $b_i \in R_n$). Indeed, for any $(P_0, \ldots, P_{n-1}) \in R_n$, let:

$$Q_i := \frac{1}{n W_i(t)} \sum_{j=0}^{n-1} \zeta_n^{-ij} P_j(t).$$

Then $Q_i \in A$ and

$$(P_1, \ldots, P_n) = \sum_{i=0}^{n-1} Q_i(t) \cdot b_i.$$

Note also that the fiber over $(0,0)$ is of degree n and contains n points. Thus it must be the spectrum of $\prod_{i=1}^n \mathbb{C}$. This means that $Y' \to X$ is étale.

Finally, Y' and Y are both étale over X, and thus Y' is étale over Y of degree 1, and thus $Y' = Y$.

This proves that every cover of X is of the form $\operatorname{Spec} R_n \to X$ with automorphism group $\mathbb{Z}/n$ (it suffices to see what are the automorphisms of the fiber over $(0,0)$ – those are cyclic permutations $(1, 2, \ldots, n)$, which take one copy of $\widetilde{X}_i$ and map it to the next copy). This means that

$$\pi_1(X, \overline{x}) = \varprojlim \mathbb{Z}/n \cong \widehat{\mathbb{Z}}.$$

Remark: usually one glues schemes along **open** subschemes, but affine schemes may be always glued along **closed** subschemes, cf. Schwede, Gluing Schemes and a Scheme Without Closed Points.

5.D **Problem:**

(a) Show that

$$\overline{\mathbb{C}((t))} = \bigcup_{n \geqslant 1} \mathbb{C}((\sqrt[n]{t})).$$

(b) Deduce that $\pi_1(Spec(\mathbb{C}((t))), \overline{x}) = \mathrm{Gal}(\overline{\mathbb{C}((t))}/\mathbb{C}((t))) \cong \widehat{\mathbb{Z}}$.

(c) Show that $\pi_1(\mathbb{A}^1_{\mathbb{C}} \backslash \{0\}, \overline{x}) \cong \widehat{\mathbb{Z}}$.

Solution:

(a) Note that $\mathbb{C}((t))$ is a complete local field. Therefore every its unramified algebraic extensions are in bijection with the algebraic extensions of the residue field (a standard fact, cf. ????). But since its residue field is algebraically closed, every extension of $\mathbb{C}((t))$ must be totally ramified. Let $K/\mathbb{C}((t))$ be a finite extension of degree n and let u_K be the uniformizer in K. Then, since it is totally ramified, $ord(u_K^n) = 1$, i.e.

$$u_K^n = t \cdot (c + \text{(higher powers)}),$$

where $c \in \mathbb{C}^*$. But by Hensel's lemma, every element $h \in \mathcal{O}_K$ such that $ord(h) = 0$ has an nth root. In particular, we can find an element $u \in \mathcal{O}_K$ such that $u^n = c + \text{(higher powers)}$. Then $(u_K/u)^n = t$ and this easily leads to $K = \mathbb{C}((t^{1/n}))$.

(b) This is immediate, since $\overline{\mathbb{C}((t))} = \bigcup_{n \geqslant 1} \mathbb{C}((t^{1/n}))$, $\mathrm{Gal}(\mathbb{C}((t^{1/n}))/\mathbb{C}) = \mathbb{Z}/n$ and $\varprojlim \mathbb{Z}/n = \widehat{\mathbb{Z}}$.

(c) Intuitively, this is clear – $\mathbb{C}((t))$ is "an infinitesimal punctured disc", and thus it should be "homotopic" to $\mathbb{A}^1_{\mathbb{C}} \backslash \{0\}$. However, I have no idea, how to make this precise. Instead, I have another solution (cf. https://math.stackexchange.com/questions/42410/finite-etale-maps-to-the-line-minus-the-origin):

let $p : Y \to X := \mathbb{A}^1_{\mathbb{C}} \backslash \{0\}$ be a connected finite étale cover. Then one can "complete" it to a cover of smooth affine curves: $p : \overline{Y} \to \overline{X} = \mathbb{P}^1_{\mathbb{C}}$. An easy application of Riemann-Hurwitz yields:

$$2g_{\overline{Y}} + \#p^{-1}(0) + \#p^{-1}(\infty) = 2,$$

and thus $g_{\overline{Y}} = 0$, $\#p^{-1}(0) = \#p^{-1}(\infty) = 1$. Therefore $\overline{Y} = \mathbb{P}^1_{\mathbb{C}}$. But the maps $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ are the rational maps, and they may be ramified at 0 and ∞ with ramification index 1 iff $p(z) = c \cdot z^n$. This map is étale over X, as can easily be seen. This easily shows that $Aut(Y/X) \cong \mathbb{Z}/n$ and

$$\pi_1(X, \overline{x}) \cong \varprojlim Aut(Y/X) \cong \varprojlim \mathbb{Z}/n \cong \widehat{\mathbb{Z}}.$$