ELEMENTS OF APPLIED MATHEMATICS

Problem set 2 - Diagonalisation and exponentials of operators

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- operator norm: $\|T\| = \max_{|x|=1} |T(x)|$

  Properties: $\|S+T\| \leqslant \|S\| + \|T\|, \quad \|ST\| \leqslant \|S\| \cdot \|T\|$

- exponential of operator: $e^T = \sum_{n \geqslant 0} T^n/n!$

  Properties: $e^S \circ e^T = e^T \circ e^S$ if $S \circ T = T \circ S$, $\quad e^{-T} = (e^T)^{-1}$, $\quad \|e^T\| \leqslant e^{\|T\|}$, $\quad \frac{d}{dt} e^{At} = Ae^{At}$, $e^{\mathrm{diag}(\lambda_1,\dots,\lambda_n)} = \mathrm{diag}(e^{\lambda_1},\dots,e^{\lambda_n})$, $\quad e^{PAP^{-1}} = Pe^AP^{-1}$.

- The Fundamental Theorem for Linear Systems

  $\dot{x} = Ax$, $x(0) = x_0$ has a unique solution given by $x(t) = e^{At}x_0$

Problem 1 Find the general solution of the equations:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} x = \dot{x} \qquad \text{and} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} y = \dot{y}$$

using diagonalisation and draw the phase portrait.

Problem 2 Write the differential equation $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$ (where $x = x(t) : \mathbb{R} \to \mathbb{R}$) in the form of the system $Ay = \dot{y}$ and solve.

Problem 3 Let the $n \times n$ matrix A have real, distinct eigenvalues. Find conditions on the eigenvalues that are necessary and sufficient for $\lim_{t\to\infty} x(t) = 0$, where $x(t)$ is any solution of $\dot{x} = Ax$.

Problem 4 Compute the operator norm of the linear transformation defined by the matrix $A = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$.

Problem 5 If T is a linear transformation on $\mathbb{R}^n$ with $\|T - I\| < 1$, prove that T is invertible and that the series $\sum_{k \geqslant 0} (I - T)^k$ converges absolutely to T^{-1}. (*Hint: Use the geometric series.*)

Problem 6 Find e^{At} and solve the linear system $\dot{x} = Ax$ for $A =$

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Show that in general $e^{A+B} \neq e^A e^B$.

Problem 7
(a) Show that if x is an eigenvector of A corresponding to the eigenvalue λ, then x is also an eigenvector of e^A corresponding to the eigenvalue e^λ.
(b) Suppose that the square matrix A has a negative eigenvalue. Show that the linear system $\dot{x} = Ax$ has at least one nontrivial solution $x(t)$ that satisfies $\lim_{t\to\infty} x(t) = 0$.
(c) Show that if A is diagonalisable then $\det e^A = e^{\mathrm{tr}\, A}$.

Problem 8 Let $T(x) = Ax$ be a linear transformation on $\mathbb{R}^n$ that leaves a subspace $E \subset \mathbb{R}^n$ invariant (i.e., for all $x \in E$, $T(x) \in E$). Show that if $x(t)$ is the solution of the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ with $x_0 \in E$, then $x(t) \in E$ for all $t \in \mathbb{R}$.