ELEMENTS OF APPLIED MATHEMATICS

PROBLEM SET 4 - JORDAN FORM AND STABILITY THEORY

- for any power series f: $$f(J_k(\lambda)) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2} & \dots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & f'(\lambda) & \dots & \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & 0 & f(\lambda) \end{bmatrix}$$
- $\mu_{k,\lambda} := \dim\ker(A-\lambda I)^k \Rightarrow \#$ (Jordan blocks corresponding to λ of size $\geqslant k$) = $\mu_k - \mu_{k-1}$,
- (minimal polynomial of A) $= \prod_{i=1}^r (x-\lambda_i)^{m_i} \Rightarrow m_i =$ size of largest Jordan block for λ_i.
- stability theory for the system $\dot{x} = Ax$:
  - stable subspace: $E^s := Span\{\Re(v), \Im(v) : v$ is a gen. eigenvector for λ with $\Re(\lambda) < 0\}$,
  - unstable subspace: $E^u := Span\{\Re(v), \Im(v) : v$ is a gen. eigenvector for λ with $\Re(\lambda) > 0\}$,
  - center subspace: $E^c := Span\{\Re(v), \Im(v) : v$ is a gen. eigenvector for λ with $\Re(\lambda) = 0\}$.

PROBLEM 1 Prove the formula for $f(J_k(\lambda))$.

(Hint: find $J_k(\lambda)^m$ by induction on m. If it's too hard, solve it for $k = 2$.)

PROBLEM 2 Find the Jordan canonical forms for the following matrices:

(a) $\begin{bmatrix} 0 & -1 & -2 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 1 & -4 \\ 4 & 3 & -5 \\ 3 & 1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 1 & 3 & 2 \\ 0 & 5 & 0 & -3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -1 & 5 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & -1 & 1 & -7 \\ 9 & -3 & -7 & -1 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 4 & -4 \end{bmatrix}$

(g) $\begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

What is the solution of the initial value problem for them?

How to describe Stable, Unstable and Center Subspaces for those matrices?

Algorithm for finding Jordan basis

1. Compute eigenvalues of the matrix A. Let λ be one of them.
2. Compute $B = A - \lambda I$, B^2, ... and $\delta_i := \dim \ker B^i$, until $\delta_n = \delta_{n+1}$.

   It's also useful to find basis for $\ker B^i$ *(relations between the columns)*.
3. Now you can draw the string diagram. The number of dots in the i-th row is $\delta_i - \delta_{i-1}$.

   You can find the part of Jordan form corresponding to λ from the diagram. A column of length k represents a Jordan block of size k.

   Each dot will correspond to a generalized eigenvector.
4. To find the vectors in the n-th row, complete in an arbitrary way basis of $\ker B^{n-1}$ to a basis of $\ker B^n$.
5. Let's find the vectors in the i-th row:
   (a) if a vector v lies in the middle of the chain above a vector w, then $v := Bw$,
   (b) to obtain the vectors that start the chain: complete in an arbitrary way basis of $\ker B^{i-1}$ and vectors obtained in (a) to a basis of $\ker B^i$.
6. Repeat for other eigenvalues. The vectors from the diagrams form a basis of generalized eigenvectors.