ELEMENTS OF APPLIED MATHEMATICS

Problem set 6 - Nonlinear systems

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1. Solve the nonhomogeneous linear system:

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{pmatrix} t \\ 1 \end{pmatrix}$$

   with the initial condition $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

2. Show by the definition that the derivative of $f(x,y) = (x\cos y, x\sin y) : \mathbb{R}^2 \to \mathbb{R}^2$ at $(0, \frac{\pi}{2})$ is a linear transformation given by the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

   How can you compute this derivative directly?

3. (a) Show that the initial value problem $x : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} \dot{x} &= |x|^{1/2} \\ x(0) &= 0 \end{aligned}$$

   has four different solutions through the point $(0,0)$. Sketch these solutions in the (t,x)-plane.

   (b) Show that $f(x) = |x|^{1/2}$ does not satisfy the Lipchitz condition in any interval $[-a,a]$, $a > 0$.

4. Let $f \in C^1(E)$ and $x_0 \in E$.

   (a) Show that there are positive constants ε and K_0 such that

$$\forall_{x,y \in N_\varepsilon(x_0) \subset E} : \qquad \|f(x) - f(y)\| \leqslant K_0 \|x - y\|.$$

   Hint. Let $K := \max_{\|x\| \leqslant \varepsilon/2} \|Df(x)\|$. *Consider the function* $g(t) = f(x + t(y-x)) : [0,1] \to \mathbb{R}^n$ *and use the formula* $g(t) = \int_0^t g'(u)\,du$.

   (b) Show that $V_a = \{u \in C([-a,a]) : \|u - x_0\| \leqslant \varepsilon\}$ is a complete space for any $a > 0$.

   (c) Consider the operator $T(u)(t) = x_0 + \int_0^t f(u(s))\,ds$.. Show that

   (i) for all $u, v \in V_a$, $\|T(u) - T(v)\| \leqslant aK_0 \|u - v\|$

   (ii) for sufficiently small a, $T : V_a \to V_a$.

   (d) Show (using the contraction principle), that the integral equation

$$u(t) = x_0 + \int_0^t f(u(t))\,dt$$

   has a unique continuous solution $u(t)$ for all $t \in [-a,a]$ provided the constant $a > 0$ is sufficiently small.