ELEMENTS OF APPLIED MATHEMATICS

Problem set 7 - Nonlinear systems, pt. 2

- maximal interval of existence,
- flow defined by an equation $\dot{x} = f(x)$,
- equilibrium point of a non-linear system: $f(x_0) = 0$
  - hyperbolic – all eigenvalues of $Df(x_0)$ have non-zero real part,
  - sink – all eigenvalues of $Df(x_0)$ have negative real part,
  - source – all eigenvalues of $Df(x_0)$ have positive real part,
  - saddle – there is at least one eigenvalue of $Df(x_0)$ with positive real part and one with negative.
- Stable Manifold Theorem: 0 – hyperbolic fixed point of $\dot{x} = f(x)$

  $Df(0)$ has k eigenvalues with negative and $n-k$ eigenvalues with positive real part

  $\Rightarrow$ there exists a k - dimensional differential manifold S (*local stable manifold*) such that:

  1. S is tangent to the stable subspace of $\dot{x} = Df(0)x$, $x(0) = 0$ at 0,
  2. $\phi_t(S) \subset S$ for $t \geqslant 0$,
  3. for all $x_0 \in S$: $\lim_{t\to\infty} \phi_t(x_0) = 0$.
- global stable manifold: $W^s(0) := \bigcup_{t\leqslant 0} \phi_t(S)$.

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1. Find the maximal interval of existence (α, β) for the following initial value problems and if $a > -\infty$ or $b < \infty$ discuss the limit of the solution as $t \to \alpha^+$ or as $t \to \beta^-$ respectively:

   (a) $\dot{x} = x^2$, $x(0) = x_0$,

   (b) $\dot{x} = \sec(x)$, $x(0) = x_0$,

   (c) $\dot{x} = x^2 - 4$, $x(0) = 0$,

   (d) $\dot{x}_1 = \frac{1}{2x_1}$, $x_1(0) = 1$, $\dot{x}_2 = x_1$, $x_2(0) = 1$.

2. Let $f \in C^1(E)$ where E is an open set in $\mathbb{R}^n$ containing the point x_0 and let $x(t)$ be the solution of the initial value problem $\dot{x}(t) = f(x(t))$ on its maximal interval of existence (α, β). Prove that if $\beta < \infty$ and if the arc-length of the half-trajectory $\Gamma_+ = \{x \in \mathbb{R}^n : x = x(t), 0 \leqslant t < \beta\}$ is finite, then it follows that the limit

$$x_1 = \lim_{t \to \beta^-} x(t)$$

exists (note that then by Corollary 2.4.1 of Perko, $x_1 \in \dot{E}$).

Hint: Assume that the above limit does not exist. This implies that there is a sequence t_n converging to β from the left such that $x(t_n)$ is not Cauchy. Use this fact to show that the arc-length of Γ_+ is then unbounded.

3. Determine the flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ for the nonlinear system $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

and show that the set $S = \{x \in \mathbb{R}^2 : x_2 = -x_1^2/4\}$ is invariant with respect to the flow ϕ_t.

4. Classify the equilibrium points (as sinks, sources or saddles) of the nonlinear system $\dot{x} = f(x)$ with $f(x)$ given by

$$\begin{bmatrix} x_2 - x_1 \\ kx_1 - x_2 - x_1 x_3 \\ x_1 x_2 - x_3 \end{bmatrix}$$

(where k is a parameter).

5. Solve the system

$$\dot{x_1} = -x_1, \qquad \dot{x_2} = x_2 + x_1^2$$

and show that S and U are given by

$$S : x_2 = -x_1^2/3, \qquad \text{and} \qquad U : x_1 = 0$$

Sketch S, U, E^s and E^u.