SFARA 2026
SEMINAR ON AUTOMORPHIC FORMS AND ABELIAN VARIETIES
ABELIAN VARIETIES

An abelian variety is, by definition, a projective algebraic variety defined over an algebraically closed field that carries the structure of an algebraic group. Elliptic curves are one-dimensional abelian varieties. An important example of an abelian variety is the Jacobian variety of a smooth projective curve. Abelian varieties over the field of complex numbers are complex tori, which can be realized as closed subvarieties of projective spaces. This seminar will be based on [2] and [1]. The classical reference is [4]. We focus mostly on complex abelian varieties.

- **Lecture 1: Abelian varieties over $\mathbb{C}$.**
  Review: algebraic varieties, morphisms, rational maps, function fields. Definition of a group variety and abelian variety ([1, p. 8]). Example: elliptic curve. Every regular map between abelian varieties is composite of a homomorphism with a translation ([1, Corollary 1.2] – you might skip the proof of the Rigidity Theorem). Abelian varieties are commutative ([1, Corollary 1.4]).

- **Lectures 2–3: Complex tori.**
  Brief "review": Lie groups and exponential maps (see e.g. [1, p. 10]). Definition of a complex torus (cf. [2, Definition 4.2]) and geometric intuition (e.g. in $\mathbb{R}^2$). Every abelian variety is a complex torus ([2, Theorem 4.1], [1, Proposition 2.1]). Review: singular cohomology (through properties and examples, rather than definitions). Cohomology of a torus ([1, p. 11]), Riemann forms, a complex torus is an abelian variety if and only if it is polarizable ([1, Theorem 2.8]).

- **Lecture 4: Isogenies.**
  Definition of an isogeny, characterization of isogenies ([1, Proposition 7.1]), degree of an isogeny, the n-torsion group $A[n]$ and the degree of the multiplication-by-n morphism over $\mathbb{C}$ ([1, Theorem 7.2]) – prove using the description of an abelian variety as a torus (cf. [2, Proposition 4.5]). Examples: multiplication by 2 on an elliptic curve and 2-torsion; division polynomials for an elliptic curve (optional); multiplication by i on $y^2 = x^3 - x$ (i.e. $\mathbb{C}/\mathbb{Z}[i]$). Define the Tate module and show that $T_\ell A \cong \mathbb{Z}_\ell^{2g}$ as an abelian group (cf. [2, Definition 10.7] – this requires the notion of an inverse system and an inverse limit).

- **Lecture 5: Line bundles.**
  Review: vector bundles, line bundles in the topological setting. Standard operations on vector bundles: pullback, pushforward, tensor product. Mention how this relates to the setting of complex geometry and algebraic geometry. Ample and very ample line bundles. The group of line bundles of a variety (Pic(A)).
  Applications to abelian varieties: theorem of the square and theorem of the cube with corollaries ([2, §1.3]) and alternative computation of the degree of $[n]$ (cf. [2, Proposition 5.17]).

- **Lectures 6–7: Dual abelian variety.**
  Dual abelian variety – first define it in the analytic setting (cf. [2, Definition 4.7]). Then define $\mathrm{Pic}^0(A)$ ([2, p. 18]) and introduce the algebraic definition with the Poincaré bundle $\mathscr{P}$ (cf. [2, Definition 6.1]). Mumford's construction of $\lambda_{\mathscr{L}}$ – [2, Definition 6.4]. Polarization – in the analytic (cf. [2, Definition 4.8]) and algebraic setting (cf. [2, Definition 6.6]). Dual homomorphism ([2, Definition 6.10]) and its properties. Rosati involution ([2, Definition 6.13]). Weil pairing ([2, Theorem 6.17] – you might show how it is defined on the level of a torus, cf. [1, Exercise 2.11]).

- **Lectures 8–9: Endomorphisms.**
  Endomorphism ring, characteristic polynomial of an endomorphism (try to sketch the proof of [1, Theorem 10.9]). Define the characteristic polynomial also through the Tate module (cf. [1, p. 46]).
  Simple abelian variety, Poincaré reducibility theorem ([2, Theorem 7.1], [1, Proposition 10.1]), definition of $End^0(A)$ and explanation, why this is a division algebra ([1, p.43]).
  Injectivity part of Tate's conjecture ([1, Theorem 10.15]) and its corollary – rank $\mathrm{Hom}(A,B) \leq 4 \dim A \cdot \dim B$, rank $\mathrm{End}(A) \leq 4(\dim A)^2$. Mention the Tate conjecture (cf. [1, Remark 10.17]). Mention Albert classification ([2, Chapter 3, Theorem 1.1]).

- **Lectures 10–11: Jacobians.**
  Divisors on curves: degree, divisor of a function (principal divisor), Picard group – [2, §9.1]. Formulate the universality property of the Jacobian (cf. [2, Proposition 9.18]) and the relation to $\mathrm{Pic}^0(C)$ (cf. [2, Theorem 9.16]). Isomorphism $H^0(C,\Omega_C) \cong H^0(J,\Omega_J)$ ([1, Proposition 2.2]) and (optionally) $H^1(C,\mathscr{O}_C) \cong H^1(J,\mathscr{O}_J)$. If time permits, sketch the construction of the Jacobian – either analytic ([3, A.6]) or through $\mathrm{Sym}^g(C)$ (cf. [3, A.8]). Example: elliptic curve is its own Jacobian.

- **Lectures 12–13: Further properties of abelian varieties.** According to the taste of the lecturer, e.g. Mordell–Weil theorem, ℓ-adic representations, Mumford–Tate conjecture, abelian varieties over fields of positive characteristic, etc.

References

[1] Milne, Abelian varieties, https://www.jmilne.org/math/CourseNotes/AV.pdf
[2] Lombardo, Abelian varieties, https://people.dm.unipi.it/lombardo/Teaching/VarietaAbeliane1718/Notes.pdf
[3] Hindry, Silverman, Diophantine geometry
[4] Mumford, Abelian varieties