Equivariant structure of the cohomologies of curves

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Cohomologies of curves

$$\underset{\text{smooth projective curve}}{X} \xrightarrow{H^1(-)} \underset{\text{fin. dim. vector space}}{H^1(X)}$$

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$$\underset{\text{smooth projective curve}}{G \curvearrowright X} \xrightarrow{\quad H^1(-) \quad} \underset{G\text{-module}}{H^1(X) \curvearrowleft G}$$

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Describe the G-**module** structure of $H^1(X)$.

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Research problem

Describe the G-**module** structure of $H^1(X)$.

Here $H^1(X)$ may mean one of the following:

$$\begin{aligned} H^1_{\mathrm{Hdg}}(X) &:= H^0(\Omega_X) \oplus H^1(\mathcal{O}_X), \\ H^1_{\mathrm{dR}}(X)&, \\ H^1_{\mathrm{cris}}(X)&. \end{aligned}$$

Over $\mathbb{C}$:

- representation theory of G comes down to characters,
- $H^1_{Hdg}(X)$ and $H^1_{dR}(X)$ can be completely described by the Chevalley–Weil formula

  (stabilizers and fundamental characters of the action).

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  (stabilizers and fundamental characters of the action).

Over $\mathbb{F}_p$ and $\mathbb{Z}_p$:

- if G has a cyclic p-Sylow subgroup:
  representation theory of G is tame,
- if G has a non-cyclic p-Sylow subgroup:
  representation theory of G is WILD.

Motivations

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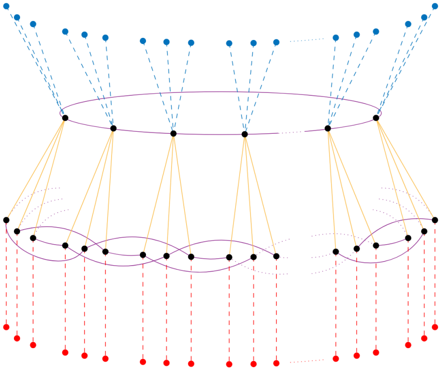
1. Classification of morphisms between curves.



Isogeny graph of elliptic curves used in the CSIDH cryptographic algorithm.

Motivations

1. Classification of morphisms between curves.

2. Lifting group actions on curves to characteristic 0.

Question

Let $X/\mathbb{F}_p$ be a curve with an action of G.
Can $G \circlearrowright X$ be lifted to characteristic 0?

Theorem (Kontogeorgis, Terezakis):
The action $G \circlearrowright X$ can be lifted if and only if there exists a lift of $G \circlearrowright H^0(X, \Omega_X)$ satisfying a certain condition.

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3. What do the representations of the group G arising from the cohomology of curves look like?

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4. $H^1_{dR}(X) \rightsquigarrow$ information about the p-torsion of $\mathrm{Jac}(X)$.

Part I:

The de Rham and Hodge cohomology

Decomposition into global and local parts

My earlier results on $H^1_{Hdg}(X)^G$ and $H^1_{dR}(X)^G$ lead me to the conjecture that

$$H^1_{dR}(X) \cong \text{global part} \oplus \text{local parts},$$
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as $k[G]$-modules where:

- global parts are the same for both cohomologies and depend only on the "shape" of the cover (stabilizers and genus of X),
- local parts depend only on the completed local rings of branch points.

HKG-covers

Setup

- G is a p-group and k is a field of characteristic p,
- $\pi : X \to Y$ is a G-cover of smooth projective curves$/k$.

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Definition

The **Harbater–Katz–Gabber cover** associated to X at $Q \in Y(k)$ is the unique G-cover $X_Q \to \mathbb{P}^1$ such that:

- it is ramified only over ∞,
- $\widehat{\mathcal{O}}_{X,Q} \cong \widehat{\mathcal{O}}_{X_Q,\infty}$ with the action of G.

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Idea

The local part of $H^1(X)$ at Q should be $H^1(X_Q)$.

Example – Klein covers

Example

$G := \mathbb{Z}/2 \times \mathbb{Z}/2$-cover of $\mathbb{P}^1$ over $\overline{\mathbb{F}}_2$:

$$X : y_0^2 + y_0 = x^3 + \frac{1}{(x-1)^7}, \qquad y_1^2 + y_1 = x^5 + \frac{1}{x^5}.$$

The HKG-curves for $0, 1$ and ∞:

$$\begin{aligned}
X_0 : & y_0^2 + y_0 = 0, & y_1^2 + y_1 = \frac{1}{x^5}, \\
X_1 : & y_0^2 + y_0 = \frac{1}{(x-1)^7}, & y_1^2 + y_1 = 0, \\
X_\infty : & y_0^2 + y_0 = x^3, & y_1^2 + y_1 = x^5.
\end{aligned}$$

The curves X_0, X_1, X_∞ are G-covers of $\mathbb{P}^1$. The idea is that they approximate $\pi : X \to Y$ around 0, 1, ∞ respectively.

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By computing explicit bases of cohomologies of X_0, X_1, X_∞ and X one can show that as $\overline{\mathbb{F}}_2[G]$-modules:

$$H^1_{dR}(X_0) \cong N_{2,0}^4$$

$$H^1_{dR}(X_1) \cong N_{2,1}^6$$

$$H^1_{dR}(X_\infty) \cong N_{2,0}^2 \oplus M_{3,1} \oplus M_{3,2}$$

$$H^1_{dR}(X) \cong N_{2,0}^7 \oplus N_{2,1}^7 \oplus M_{3,1} \oplus M_{3,2}$$

where $N_{2,i}$, $M_{3,i}$ – indecomposable $\overline{\mathbb{F}}_2[G]$-modules of dimensions 2, 3.

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Case of a general p-group G

Theorem (JG, in progress)

With the above setup, as $k[G]$-modules:

$$H^1_{dR}(X) \cong k[G]^{2(g_Y - d)} \oplus I_{X/Y} \oplus I_{X/Y}^{\vee} \oplus \bigoplus_{Q \in Y(k)} H^1_{dR}(X_Q),$$

where:

- g_Y is the genus Y,
- X_Q is the HKG curve approximating X above Q,
- $I_{X/Y} := \ker \left(\mu \oplus \sum : k[G]^d \oplus \bigoplus_Q I_{G,G_Q} \to k[G]\right)$,
- $I_{G,H} := \mathrm{Ind}_H^G I_H$,
- group of the largest étale subcover $X_{et} \to Y$ is generated by d elements.

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$$\mathrm{Indec}^{\star}(k[G]) := \left\{ \begin{array}{c} \text{indecomposable summands of } H^1_{\star}(X) \\ \text{for curves } X/k \text{ with a } G\text{-action} \end{array} \right\}.$$

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Corollary (JG, [6], 2024)

Let $p > 2$. If the p-Sylow of G is not cyclic, then

$$\mathrm{Indec}^{Hdg}(k[G]) \quad \text{and} \quad \mathrm{Indec}^{dR}(k[G])$$

are infinite.

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Idea: $(\mathbb{Z}/p \times \mathbb{Z}/p)$-covers $z_0^p - z_0 = x^m$, $z_1^p - z_1 = a \cdot x^m$.

Part II:

Crystalline cohomology

Crystalline cohomology

Definition

Crystalline cohomology of a curve X over $\mathbb{F}_p$ is the de Rham cohomology of any lift $\mathbf{X}$ of X to $\mathbb{Z}_p$:

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Applications:

counting points on varieties over $\mathbb{F}_p$ (Kedlaya's algorithm), supersingularity, Galois representations (p-adic Hodge theory)

Cyclic p-Sylow subgroup results

Theorem (A. Kontogeorgis & JG, 2025)

Let G have cyclic p-Sylow and act on $X/\mathbb{F}_p$.
The $\mathbb{F}_p[G]$-module $H^1_{dR}(X)$ is determined by g_X and:

- the higher ramification groups of the action,
- the fundamental characters of the action.

Cyclic p-Sylow subgroup results

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Main tool: a $\mathbb{Z}_p[G]$-module M, finite free over $\mathbb{Z}_p$, is determined by the diagram:

$$H^1(F_1, M) \leftrightarrows H^1(F_2, M) \leftrightarrows \ldots \leftrightarrows H^1(F_n, M)$$

(where $F_i := \mathbb{Z}/p^i$) up to "trivial summands". [Yakovlev, 1970]

Formulas for $G = \mathbb{Z}/p^n$

$$H^1_{dR}(X) \cong J_{p^n}^{2(g_Y-1)} \oplus J^2_{p^n - p^{n-m}+1} \oplus \bigoplus_{\substack{Q \in B \\ Q \neq Q_0}} J^2_{p^n - p^n/e_Q}$$

$$\oplus \bigoplus_{Q \in B} \bigoplus_{t=0}^{m_Q - 1} J_{p^n - p^{n+t}/e_Q}^{u_Q^{(t+1)} - u_Q^{(t)}},$$

where:

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where:

- J_i – representation of $G = \mathbb{Z}/p^n$ over $\mathbb{F}_p$ given by $i \times i$-Jordan block with eigenvalue 1,
- e_Q – ramification indices,
- $u_Q^{(i)}$ – upper ramification jumps,
- p^m – maximal ramification index.

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where:

- $\mathbb{J}_{p^n}, \mathbb{J}_{p^n-p^{n-m}+1}, \mathbb{J}_{p^n-p^i}$ – explicit representations of $G = \mathbb{Z}/p^n$ over $\mathbb{Z}_p$,
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- global part,

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Question

How about $H^1_{\mathrm{cris}}(X)$ for other groups G?

Thank you for your attention!

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Backup slides

Example:
Crystalline cohomology mod 9 of the curve $y^2 = x^3 - x$ over $\mathbb{F}_3$:

$$\left\{ \left(\left[\frac{y}{x^3-x}\right] dx + V\left(\frac{-x^2-2}{x^4y-2x^2y+y}\,dx\right) + dV\left(\frac{x^{18}y - x^{16}y + x^{14}y + x^8y - x^6y + x^4y}{x^4-2x^2+1}\right)\right), \right.$$

$$V\left(x^{14}y + x^{12}y + 2x^{10}y - 2x^8y - x^6y + x^4y + 2x^2y - y\right),$$

$$\left. \left[\frac{y}{x^3-x}\right] dx + V\left(\frac{-x^2-2}{x^4y-2x^2y+y}\,dx\right) + dV\left(\frac{x^2y+y}{x^4-2x^2+1}\right) \right\},$$

$$\left\{ \left(\left[-\frac{y}{x^2-1}\right] dx + V\left(\frac{2x^7-2x^5+2x^3+x}{x^4y-2x^2y+y}\,dx\right) + dV\left(\frac{-x^{23}y + x^{21}y - x^{19}y - x^{13}y + x^{11}y - x^9y}{x^4-2x^2+1}\right)\right), \right.$$

$$\left[-\frac{2y}{x}\right] + V\left(\frac{-x^{20}y - x^{18}y - 2x^{16}y + 2x^{14}y + x^{12}y - x^{10}y - 2x^8y + x^6y - x^4y + 2x^2y - y}{x}\right),$$

$$\left. \left[\frac{y}{x^4-x^2}\right] dx + V\left(\frac{2x^6+x^4+2x^2-2}{x^7y-2x^5y+x^3y}\,dx\right) + dV\left(\frac{2xy}{x^4-2x^2+1}\right) \right\}$$

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- **Theorem (Nakajima, 1984)**

  If $X \to Y$ is an étale G-cover then:

$$H^0(X, \Omega_X) \cong k[G]^{(g_Y - d)} \oplus \ker(k[G]^d \twoheadrightarrow I_G).$$

The decomposition for a generic curve:

- **Step I:** find a function z with low order poles, such that $\{g^*(z) : g \in G\}$ is linearly independent.
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- **Step III:** prove inductively a formula for cohomologies of an HKG-cover.

Explicit form of the summands of crystalline cohomology

$$\mathbb{J}_{p^n-p^{n-i}} := \ker(\mathbb{Z}_p[G] \to \mathbb{Z}_p[G/G_i]),$$
$$\mathbb{J}_{p^n-p^{n-i}+1} := \ker(\mathbb{Z}_p[G] \to \mathbb{Z}_p[G/G_i]/\mathbb{Z}_p).$$

where W is the image of the map
$N_{G/G_i} := \sum_{g\in G/G_i} g : \mathbb{Z}_p \to \mathbb{Z}_p[G/G_i]$.