PURITY THEOREM

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1. Introduction

Notation: X_0 – over $k = \mathbb{F}_q$, of dimension r. $X = (X_0)_{\overline{k}}$.
Suppose that X_0 is smooth and projective. Recall that, as shown on the last lecture:

$$Z(X_0, t) = \prod_{i=0}^{2r} P_i(t)^{(-1)^i}, \text{ where } P_i(t) = \det(I - Ft | H^i(X, \mathbb{Q}_\ell)) = \prod_j (1 - \alpha_{ij} t) \in \mathbb{Z}[t].$$

The remaining part of Weil's conjecture is:

Theorem 1 (Weil's Riemann hypothesis)**.** *X_0 – smooth and projective, λ – an eigenvalue of F on H^i $\Rightarrow$all complex conjugates of λ have absolute value $q^{i/2}$.* [1]

There are two proofs due to Deligne:

- by induction, using Lefschetz fibration,
- by "generalizing" it to purity theorem.

2. Purity theorem for proper and smooth morphisms

Note that $H^i(X, \mathbb{Q}_\ell) = (R^i f_{0,*}(\mathbb{Q}_\ell))_{\overline{k}}$. This suggest a more general version for proper and smooth $f : X \to S$. In order to formulate it we need:

Definition. *A constructible $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}_0$ on a k-scheme X_0 is **pure of weight** i if $\forall_{x \in X_0 \text{ – closed}} \forall_{\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}}$ every $\mathbb{Q}_\ell$-eigenvalue λ of F_x on $\mathcal{F}_x$ satisfies:*

$$|\iota(\lambda)| = (\#\kappa(x))^{i/2}.$$

Remark. *If it this holds for a fixed ι, $\mathcal{F}$ is ι-pure.*

Remark. *If a k-vector space has an action of Frobenius with eigenvalues as above, we will also say that it is pure of weight w.*

Properties of pure sheaves: $\mathcal{F}_0$, $\mathcal{G}_0$ are pure of weight $w_1, w_2 \Rightarrow$

- $\mathcal{F}_0^\vee$ – pure of weight $-w_1$,
- $\mathcal{F}_0 \otimes \mathcal{G}_0$ – pure of weight $w_1 + w_2$,
- $\mathcal{F}_0(d)$ – pure of weight $w_1 - 2d$.

Theorem 2 (purity for proper and smooth morphisms)**.** *Let $f_0 : X_0 \to Y_0$ – a proper and smooth morphism of $k = \mathbb{F}_q$-varieties. Let $\mathcal{F}_0$ – a pure sheaf on X_0 of weight i. Then, for all j, $R^j f_{0,*}\mathcal{F}_0$ is pure of weight $i + j$.*

Note that this implies Weil's Riemann Theorem for $Y_0 = \operatorname{Spec} k$, $\mathcal{F}_0 = \mathbb{Q}_\ell$.

[1](α_{ij} is a q-Weil number of weight i)

3. Purity for non-proper morphisms

We will suppose now that X_0 is arbitrary. Recall that then:

$$P_i(t) = \det(I - Ft | H^i_c(X, \mathbb{Q}_\ell)).$$

Example. *Let E_0 be an elliptic curve over k, $Z_0 = \{p_1, p_2\} \overset{i_0}{\hookrightarrow} E_0$, $X_0 = E_0 \setminus Z_0 \overset{j_0}{\hookrightarrow} E_0$. Then:*

$$0 \to (j_0)_!(\mathbb{Z}/\ell^n) \to \mathbb{Z}/\ell^n \to (i_0)_*(\mathbb{Z}/\ell^n) \to 0,$$

which gives:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{H^0(Z,\mathbb{Q}_\ell)}{H^0(E,\mathbb{Q}_\ell)} & \longrightarrow & H^1_c(U, \mathbb{Q}_\ell) & \longrightarrow & H^1(E, \mathbb{Q}_\ell) & \longrightarrow & 0 \\ & & \downarrow \wr & & & & \downarrow \wr & & \\ & & \frac{\mathbb{Q}_\ell(0) \oplus \mathbb{Q}_\ell(0)}{\mathbb{Q}_\ell(0)} \cong \mathbb{Q}_\ell(0) & & & & \textit{weight 1 by purity} & & \end{array}$$

and thus $H^1_c(U, \mathbb{Q}_\ell)$ is not pure.

Definition. *$\mathcal{F}$ is **mixed** with weights $w_1, \ldots, w_n$, if it has a finite increasing filtration by constructible $\mathbb{Q}_\ell$-subsheaves with quotients pure of weights $w_1, \ldots, w_n$.*

Theorem 3 (Deligne's purity theorem)**.** *Let $f_0 : X_0 \to Y_0$ – a proper and smooth morphism of k-varieties. Let $\mathcal{F}_0$ – a mixed sheaf on X_0 of weights $\leq i$. Then, for all j, $R^j(f_0)_! \mathcal{F}_0$ is mixed of weights $\leq i + j$. Moreover, each weight of $R^w(f_0)_! \mathcal{F}_0$ is congruent mod $\mathbb{Z}$ to a weight of $\mathcal{F}_0$.*

Corollary 4 (Riemann's hypothesis for general varieties)**.** *X – a variety over k $\Rightarrow H^i_c(X, \mathbb{Q}_\ell)$ is mixed with weights $\leq i$, i.e.*

$$\forall_\sigma |\sigma(\alpha_{ij})| = q^{l/2} \quad \textit{for some } l = l_{ij} \in \mathbb{Z}, l \leq i.$$

Corollary 5 (RH for proper smooth varieties)**.** *X_0 – smooth and proper $\Rightarrow H^i(X, \mathbb{Q}_\ell)$ is pure of weight i.*

Proof. By Poincaré duality:

$$H^i(X, \mathbb{Q}_\ell) \cong H^{2r-i}(X, \mathbb{Q}_\ell(r))^\vee.$$

Deligne purity implies that $H^i(X, \mathbb{Q}_\ell)$ is mixed of weight $\leq i$ and $H^{2r-i}(X, \mathbb{Q}_\ell(r)) = H^{2r-i}(X, \mathbb{Q}_\ell)(r)$ is mixed of weight $\leq (2r - i) - 2r = -i$. Thus $H^i(X, \mathbb{Q}_\ell)$ is pure of weight i. □

Another applications:

- hard Lefschetz theorem: (not proven today)

  Theorem 6. *The map:*

  $$c_1(H)^n : H^{2r-n}_{et}(X, \mathbb{Q}_\ell) \to H^{2r+n}_{et}(X, \mathbb{Q}_\ell)$$

  (n-fold cup product by the cohomology class of hyperplane section) is an isomorphism.

- Lang–Weil estimates.

4. Lang–Weil estimates

Idea: by Riemann's hypothesis:

$$\# X(\mathbb{F}_q) = \sum_{i=0}^{2r} \mathrm{tr}(F | H^i) \approx q^r + (\sum_{i<2r} \dim_k H^i) \cdot q^{r-1/2}$$

Theorem 7 (Lang–Weil estimates)**.** *Let $X \subset \mathbb{P}^n_k$ be irreducible of dimension r and degree d. Then:*

$$|\# X(k) - q^r| \leq (d-1)(d-2) q^{r-1/2} + A(n, d, r) q^{r-1}.$$

Lemma 8 ("weak Lang-Weil")**.** $\# X(k) \leq A_1(n, d, r) q^r$

Proof. Induction on n, using:

$$|X(k)| \leq \sum_{\lambda \in \mathbb{P}^1} |(X \cap H_\lambda)(k)|,$$

where $H_\lambda = \{x_0 = \lambda x_1\}$. □

Lemma 9. $\#\{H \subset \mathbb{P}^n_k$ – *k-hyperplane* : $X \cap H$ *is not geo. irreducible or not gen. reduced* $\} \leq A_2(n,d,r) \cdot q^{n-1}$

Proof of Lang-Weil. For $r = 0$ – easy, for $r = 1$ – we take the normalization of X and apply Riemann's hypotheses to it (exercise!).
$T(r-1) \Rightarrow T(r)$: we use induction on n. Let:

$$W := \{(x, H) \in X \times (\mathbb{P}^n_k)^* : x \in H\}.$$

Then, using $\mathrm{pr}_1 : W \to X$:

$$\#W(\mathbb{F}_q) = \#\mathbb{P}^{n-1}(\mathbb{F}_q) \cdot \#X(\mathbb{F}_q)$$

and on the other hand, using $\mathrm{pr}_2 : W \to (\mathbb{P}^n_k)^*$:

$$\#W(\mathbb{F}_q) = \sum_H |(X \cap H)(\mathbb{F}_q)| = \sum_{H_1} |(X \cap H_1)(\mathbb{F}_q)| + \sum_{H_2} |(X \cap H_2)(\mathbb{F}_q)|$$

where H_1's are such that $X \cap H_1$ is not geometrically irreducible or not generically reduced.

- By Lemma 9: $\#\{H_1\} = |\#(\mathbb{P}^n)^* - \#\{H_2\}| \leq A_2 \cdot q^{n-1}$,
- By Lemma 8: $|(X \cap H_1)(\mathbb{F}_q)| \leq d \cdot A_1 q^{r-1}$,
  (note that the number of irreducible components of $(X \cap H_1)_{red}$ is $\leq d$, and each has degree $\leq d$.)
- By induction on n: $|(X \cap H_2)_{red}(\mathbb{F}_q) - q^{r-1}| \leq (d-1) \cdot (d-2) q^{r-3/2} + A(n-1, d, r-1) \cdot q^{r-2}$.

□

Proof of Lemma 9. Let $V \subset \mathbb{P}^n$ be a closed subvariety of dimension r. Consider the set:

$$\{(H_1, \ldots, H_{r+1}) \in ((\mathbb{P}^n)^*)^{\times(r+1)} : \bigcap_i H_i \cap V \neq \varnothing\}.$$

One can show it is a hypersurface $Z(R_V)$ in $(\mathbb{P}^{n*})^{r+1}$ (where R_V is a polynomial in $(r+1)$ sets of $(n+1)$ variables), homogeneous of degree d in each set of variables).

Definition. *R_V is the* ***Cayley form*** *of V. Coefficients of R_V are the* ***Chow coordinates*** *of V, denoted $c(V) \in \mathbb{P}^M$, where $M = (n+1) \cdot (r+1) - 1$.*

Definition. *If $D = \sum_i n_i V_i$, then $R_D := \prod_i R_{V_i}^{n_i}$, $c(D) =$ coefficients of R_D.*

Proof of Lemma 9. We want to estimate the cardinality of:

$$R := \{H \in (\mathbb{P}^n)^* : H \cap R \text{ is not a variety}\}.$$

Let:

$$\begin{aligned} C &:= \{c(D) : D \text{ is a cycle of dim. } r \text{ and degree } d, \text{ which is not a variety }\} \\ &= Z(\phi_1, \ldots, \phi_s) \subset \mathbb{P}^M, \end{aligned}$$

where $\phi_i \in \mathbb{F}_p[...]$ depend only on n, r and d. Note that $R_{X \cap H}(\ldots) = R_X(\ldots, H)$ and thus $c_X(H) := c(X \cap H)$ is a tuple of forms of degree d in the "variable" $H \in (\mathbb{P}^n)^*$. Then:

$$R \subset Z(\phi_1 \circ c_X, \ldots, \phi_s \circ c_X)$$

and we can use Lemma 8 for each of hypersurfaces $Z(\phi_i \circ c_X)$ (dimension $= n-1$, degree $= \deg \phi_i$). □

5. About the proof of purity theorem

Very general outline:

(1) Prove purity for real sheaves (i.e. characteristic polynomial of Frobenius has real coefficients).
(2) Reduce to the purity of $H^1(\mathbb{P}^1, j_! \mathcal{F})$, where $\mathcal{F}_0$ – sheaf on U_0, $j_0 : U_0 \subset \mathbb{P}^1$.
(3) Use Fourier transform – it is real (and thus pure) and its stalk is $H^1(\mathbb{P}^1, j_! \mathcal{F})$.

6. Fourier transform

With every $\mathbb{Q}_\ell$-constructible sheaf $\mathcal{F}_0$ on X_0 we can associate the function

$$f^{\mathcal{F}_0} : X(\mathbb{F}_{q^n}) \to \mathbb{C}, \quad f^{\mathcal{F}_0}(x) := \operatorname{tr}(F_x | (\mathcal{F}_0)_x)$$

for every n. We can define also f^{K_0} for $K_0 \in D^b_c(X_0, \mathbb{Q}_\ell)$ as:

$$f^{K_0} := \sum_i (-1)^i f^{\mathcal{H}^i K_0}.$$

Then:

(1) $f^{\mathcal{F}_0 \otimes \mathcal{G}_0} = f^{\mathcal{F}_0} \cdot f^{\mathcal{G}_0}$,
(2) $f^{g^* \mathcal{K}_0} = f^{\mathcal{K}_0} \circ g$,
(3) $f^{\mathbf{R} g_! \mathcal{K}_0}(x) = \sum_{y \in X_x(\mathbb{F}_{q^n})} f^{\mathcal{K}_0}(y)$.

Remark (optional)**.** *If $\mathcal{F}$ is lisse and semisimple, then $f^{\mathcal{F}}$ determines $\mathcal{F}$, because the Frobenii are dense in the monodromy group.*

From now on: $X_0 = \mathbb{A}^1$. Let $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}_\ell}^\times$ be a fixed non-trivial additive character. Note that ψ induces a map $\mathbb{F}_{q^n} \xrightarrow{\operatorname{tr}} \mathbb{F}_q \xrightarrow{\psi} \overline{\mathbb{Q}_\ell}^\times$, which we will denote also by ψ.

Question. *How to construct a sheaf $\mathcal{F}_0$ with $f^{\mathcal{F}} = \psi$?*

Definition. *The Artin-Schreier sheaf $\mathcal{L}_0(\psi)$ on $\mathbb{A}^1_0$ is the sheaf corresponding to the representation:*

$$\pi_1(\mathbb{A}^1_0, \overline{x}) \twoheadrightarrow \mathbb{F}_q \xrightarrow{\psi} \overline{\mathbb{Q}_\ell}^\times,$$

where the first map comes from the finite étale cover:

$$\mathbb{A}^1_0 = \operatorname{Spec} \mathbb{F}_q[y] \to \operatorname{Spec} \mathbb{F}_q[x], \qquad x \mapsto y^q - y.$$

Lemma 10. *$f^{\mathcal{L}_0(\psi)}(x) = \psi^{-1}(x)$.*

Proof. Note that the action of π_1 on $\mathcal{L}_0(\psi)_{\overline{x}}$ is (by definition) given by the above homomorphism $\pi_1 \to \overline{\mathbb{Q}_\ell}^\times$. Let σ be the arithmetic Frobenius and let $x \in \mathbb{A}^1_0(\mathbb{F}_{q^n})$. We want to compute its image via:

$$\pi_1(\mathbb{A}^1_0, \overline{x}) \to \mathbb{F}_q$$

(i.e. in the Galois group of Artin–Schreier cover). Note that $\sigma(x) := x^{q^n}$, $\sigma(y) := y^{q^n}$. Thus:

$$\begin{aligned} y^q &= y + x \\ y^{q^2} &= y^q + x^q = y + x + x^q \\ &\dots \\ y^{q^n} &= y + x + x^q + \ldots + x^{q^{n-1}} = y + \operatorname{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x) \end{aligned}$$

and σ maps to translation by $\operatorname{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x)$. Thus σ acts by multiplication by $\psi(\operatorname{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x)) =: \psi(x)$ on the stalk and F_x by $\psi^{-1}(x)$. □

Definition. *Define the **Fourier transform** of $f : \mathbb{F}_{q^n} \to \overline{\mathbb{Q}_\ell}^\times$ with respect to ψ:*

$$FT_\psi(f) : \mathbb{F}_{q^n} \to \overline{\mathbb{Q}_\ell}^\times, \quad FT_\psi(f)(x) := \sum_{y \in \mathbb{F}_{q^n}} f(y)\psi(-xy).$$

Question. *Given $\mathcal{F}_0$, how to construct sheaf $FT_\psi(\mathcal{F}_0)$ satisfying:*

$$f^{FT_\psi(\mathcal{F}_0)} = FT_\psi(f^{\mathcal{F}_0})?$$

Definition.

$$FT_\psi(\mathcal{K}_0) := \boldsymbol{R}\operatorname{pr}_{1,!}(\operatorname{pr}_2^* \mathcal{K}_0 \otimes m^* \mathcal{L}_0(\psi))[1]$$

where $\operatorname{pr}_i : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ and $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the multiplication.

Properties:

(1) if $\mathcal{K}_0 = \mathcal{F}_0$ is a sheaf, $FT_\psi(\mathcal{F}_0)$ is a complex with cohomology concentrated at most in degree $-1, 0, 1$,
(2) $f^{FT_\psi(\mathcal{F}_0)} = -FT_\psi(f^{\mathcal{F}_0})$.
(3) $(FT_\psi(\mathcal{F}_0))_{\overline{a}} = H^1_c(\mathbb{A}^1, \mathcal{F} \otimes \mathcal{L}(\psi_a))$, where $\psi_a(x) := \psi(a \cdot x)$.

Proof. (2)

$$\begin{aligned} f^{FT_\psi(\mathcal{F}_0)}(x) &= -f^{\mathbf{R}\,\mathrm{pr}_{1,!}\left(\mathrm{pr}_2^* \mathcal{K}_0 \otimes m^* \mathcal{L}_0(\psi)\right)}(x) = \\ &= -\sum_{(y,x)\in(\mathbb{A}^1\times\mathbb{A}^1)(\mathbb{F}_{q^n})} f^{\mathrm{pr}_2^* \mathcal{K}_0 \otimes m^* \mathcal{L}_0(\psi)}(y,x) \\ &= -\sum_{(y,x)\in(\mathbb{A}^1\times\mathbb{A}^1)(\mathbb{F}_{q^n})} f^{\mathrm{pr}_2^* \mathcal{K}_0}(y,x) \cdot f^{m^* \mathcal{L}_0(\psi)}(y,x) \\ &= -\sum_{(y,x)\in(\mathbb{A}^1\times\mathbb{A}^1)(\mathbb{F}_{q^n})} f^{\mathcal{K}_0}(x) \cdot f^{\mathcal{L}_0(\psi)}(x \cdot y) \\ &= -\sum_{(y,x)\in(\mathbb{A}^1\times\mathbb{A}^1)(\mathbb{F}_{q^n})} f^{\mathcal{K}_0}(x) \cdot \psi^{-1}(x \cdot y). \end{aligned}$$

□

Some definitions

- constructible sheaves:
  - $\mathcal{F}$ on X_{et} is constructible if it is locally constant in etale topology and has finite stalks,
  - $\mathcal{F}$ is $\mathbb{Z}_\ell$-constructible if $\mathcal{F} = (\mathcal{F}_n)$, where $\mathcal{F}_n$ – constructible $\mathbb{Z}/\ell^n$-module,
  - category of $\mathbb{Q}_\ell$-constructible sheaves – $\mathbb{Z}_\ell$-constructible sheaves with Hom's tensored by $\mathbb{Q}_\ell$, (analogously we define E-constructible sheaves for $[E : \mathbb{Q}_p] < \infty$)
  - category of $\overline{\mathbb{Q}_\ell}$-constructible sheaves – E-constructible sheaves for all $[E : \mathbb{Q}_p] < \infty$ with Homs:

    $$\mathrm{Hom}_{\overline{\mathbb{Q}_\ell}}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_F(\mathcal{F} \otimes_E F, \mathcal{G} \otimes_E F)$$

    (where $\mathcal{F}$ is E-constr. and $\mathcal{G}$ is E'-constr. and $F \supset E, E'$ is a finite field extension)
- $N_m = \sum_{r=0}^{2d} \mathrm{tr}(F^m | H^r_c)$
- cohomology with compact support:
  - extension by zero: $j : U \hookrightarrow X$ – open embedding $\Rightarrow j_! \mathcal{F}$ – sheaf associated with

    $$(\phi : V \to X) \mapsto \begin{cases} \mathcal{F}(V), & \varphi(V) \subset U \\ 0, & \text{otherwise.} \end{cases}$$

  - $H^i_c(X, \mathcal{F}) := H^i(X, j_! \mathcal{F})$, where $j_0 : X_0 \hookrightarrow X'_0$ is an open embedding with dense image and X'_0 is proper over k.
  - $f : X \to S \Rightarrow R^i f_! \mathcal{F} := R^i f'_*(j_! \mathcal{F})$, where $j : X \hookrightarrow X'$ – open embedding with dense image into a proper S-scheme X'
- $D^b_c(\mathbb{A}^1_0, \overline{\mathbb{Q}_\ell})$ is the "derived" category of bounded complexes of étale $\overline{\mathbb{Q}_\ell}$-sheaves on $\mathbb{A}^1_0$ with constructible cohomology sheaves

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