PURITY THEOREM

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1. Problems

(1) Let $X \subset \mathbb{P}^n$ be a curve of degree d over $\mathbb{F}_q$ (not necessarily smooth). Prove that:

$$|\#X(\mathbb{F}_q) - q| \leq (d-1)\cdot(d-2)\cdot q^{1/2} + \frac{d(d-1)(d-2)}{2} + 1.$$

Hints:

(a) Let $\pi : Y \to X$ be the normalization of (the projective closure of) X. Find the relation between $\#X(\mathbb{F}_q)$ and $\#Y(\mathbb{F}_q)$. Use Weil estimates for Y.

(b) Let $\phi : X_{\overline{k}} \to \mathbb{P}^2_{\overline{k}}$ be a projection with image C (degree d plane curve with at most double points) and $\psi = \phi \circ \pi_{\overline{k}}$. Show that $\sum_{x \in (X_{\overline{k}})_{sing}} \deg(\pi_{\overline{k}}^{-1}(x)) \leq \sum_{x \in C_{sing}} \deg(\psi^{-1}(x))$.

(c) Estimate $\deg(\psi^{-1}(x))$ and show that $\#C_{sing} \leq \frac{(d-1)(d-2)}{2}$, using that $\psi_* \mathcal{O}_{Y_{\overline{k}}}/\mathcal{O}_C$ is a sheaf with a finite support.

(2) Let V be an r-dimensional variety in $\mathbb{P}^n$.

(a) Show that the set

$$W := \{(H_1, \ldots, H_{r+1}) \in ((\mathbb{P}^n)^*)^{\times(r+1)} : \bigcap_i H_i \cap V \neq \varnothing\}$$

is of codimension 1 in $((\mathbb{P}^n)^*)^{\times(r+1)}$.

(Hint: consider the incidence variety $\{(H_1, \ldots, H_{r+1}, x) \in ((\mathbb{P}^n)^*)^{\times(r+1)} \times X : x \in \bigcap_i H_i\}$ – compute its dimension by projecting and compare it with the dimension of W)

(b) Show that W determines V.

(Hint: give a criterion for $x \in V$ in terms of elements of W)

(3) Show that for $f : \mathbb{A}^1(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell$:

$$(FT_\psi \circ FT_{\psi^{-1}})f = q^n \cdot f.$$

(4) Show that if we base change $\mathcal{L}_0(\psi)$ to $\mathbb{F}_{q^n}$, we get $\mathcal{L}_0(\psi \circ \mathrm{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q})$.

(5) Let $\wp : \mathbb{A}^1 \to \mathbb{A}^1$, $y \mapsto y^p - y$ be the Artin–Schreier cover. Let also $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell$ be a non-trivial (additive) character and $\psi_x(y) := \psi(x \cdot y)$. Show that:

$$\wp_*(\overline{\mathbb{Q}}_\ell) = \bigoplus_{x \in \mathbb{F}_q} \mathcal{L}_0(\psi_x).$$

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